# Quantum Symmetries of Combinatorial Structures: A Computer Assisted Approach

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## Abstract

In this thesis, we study the quantum symmetries of various combinatorial objects. More precisely, we investigate the quantum automorphism groups of vertex-transitive graphs, which were first defined by Banica and Bichon in [4], [13], and introduce the new notions of quantum switching isomorphisms of signed graphs and quantum automorphism groups of matroids.

The results include a complete determination of the existence of quantum symmetries of all vertex-transitive graphs on 12 vertices, extending work done in [6], [22], [23], and the computation of the quantum automorphism groups of two of these graphs aided by a Gröbner basis computation.

Inspired by the graph isomorphism game introduced in [2], we moreover constructed a switching isomorphism game for signed graphs. Analogously to the study of the graph isomorphism game in [51], we find that there is a strong connection between the perfect quantum strategies for the switching isomorphism game and the hyperoctahedral quantum group. However, we also find that for connected signed graphs, any true quantum switching isomorphism always implies the existence of a true quantum isomorphism of the underlying graphs.

Lastly, we define several quantum automorphism groups for matroids. It turns out that unlike the classical automorphism groups of matroids, the quantum automorphism groups depend on the axiom system chosen for the matroid. We then study the quantum automorphism groups of several matroids, using amongst others Gröbner basis computations.

This work is based in parts on the research articles [26], [73] of which the author of the thesis was the author and a coauthor respectively.

## Zusammenfassung


In dieser Arbeit untersuchen wir die Quantensymmetrien verschiedener kombinatorischer Objekte. Um genauer zu sein, befassen wir uns mit den Quantenautomorphismengruppen von vertextransitiven Graphen, die zuerst von Banica und Bichon in [4], [13] definiert wurden, und führen die neuen Begriffe der Quantenswitchingisomorphie von signierten Graphen und der Quantenautomorphismengruppen von Matroiden ein.

Unter den Ergebnissen ist eine vollständige Bestimmung der Existenz von Quantensymmetrien von allen vertextransitiven Graphen auf 12 Punkten. Dies erweitert die Arbeit aus [6], [22], [23]. Weiterhin berechnen wir die Quantenautomorphismengruppen von zwei dieser Graphen, wobei teilweise auf Gröbnerbasisberechnungen zurückgegriffen wird.

Außerdem konstruieren wir ein Switching-Isomorphie-Spiel für signierte Graphen nach dem Vorbild des Graph-Isomorphie-Spiels aus [2]. Analog zur Analyse des Graph-Isomorphie-Spiels in [51] können wir zeigen, dass es eine starke Verbindung zwischen perfekten Quantenstrategien für das Switching-Isomorphie-Spiel und der hyperoktaedrische Quantengruppe. Wir können allerdings auch zeigen, dass für zusammenhängende signierte Graphen jeder wahre Quantenswitchingisomorphismus schon die Existenz eines wahren Quantenswitchingisomorphismus der zugrunde liegenden Graphen impliziert.

Schließlich definieren wir verschiedene Quantenautomorphismengruppen für Matroide. Es stellt sich heraus, dass – anders als die klassische Automorphismengruppen von Matroiden – die Quantenautomorphismengruppen von dem Axiomsystem abhängt, welches man wählt um den Matroid zu beschreiben. Wir untersuchen die Quantenautomorphismengruppen von verschiedenen Matroiden, unter anderem mithilfe von Gröbnerbasisberechnungen.

Diese Arbeit basiert teilweise auf den Forschungsartikeln [26], [73], welche der Autor der Dissertation verfasst bzw. mitverfasst hat.

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# Contents

# Introduction

The subject of this thesis is the study of quantum symmetries of certain combinatorial structures, namely simple graphs, signed graphs and matroids. These symmetries are either studied in the sense of Woronowicz's compact matrix quantum groups or as the perfect quantum strategies of certain nonlocal games.

We now give a brief overview of the history and important results on quantum automorphism groups and nonlocal games and then state the main results of the thesis.

## Quantum automorphism groups

**Compact matrix quantum groups.** When studying symmetry, one typically studies groups and their actions. However, with the discovery of quantum effects it has become clear that one needs to study a bigger class of objects. In this context, quantum groups were defined in 1986 by Drinfeld [33] and Jimbo [44]. Shortly afterwards, Woronowicz [89] defined compact quantum groups, where the idea is that instead of studying a compact group $G$ with its binary operation $\circ : G \times G \to G$, one studies its algebra of continuous functions $C(G)$ and the map $\Delta : C(G) \to C(G \times G) \cong C(G) \otimes C(G)$.

For any compact group $G$, the function algebra $C(G)$ will be a commutative $C^*$-algebra, and by Gelfand duality, any commutative $C^*$-algebra is of the form $C(X)$ for a compact space $X$. Therefore, in order to generalise the concept of a compact group, one takes a no longer necessarily commutative $C^*$-algebra $A = C(G)$ together with a comultiplication map $\Delta$ to define a compact quantum group. Here, the underlying compact group $G$ only exists when $C(G)$ is commutative and is only symbolical otherwise. Woronowicz moreover defined compact matrix quantum groups as a part of compact quantum groups that generalise the classical compact matrix groups. In this thesis, all compact quantum groups that are studied will be compact matrix quantum groups.

Using this framework, Wang captured quantum symmetries of finite spaces by defining the quantum permutation group $S_n^+$ as the quantum analogue of the permutation group $S_n$ in [82]. The name quantum permutation group is justified, since $S_n^+$ is the quantum automorphism group of $n$ points in the sense that it acts maximally on this set. With this in mind, one can then ask what happens if one adds more structure to these points and asks that the structure be respected by the action.

**Graphs.** Graphs are a useful tool in modeling many things, from transportation networks over disease spreading to the data used in machine learning algorithms. As such, their study and amongst others the study of their symmetries has been a subject in mathematics for a long time. In its simplest form, a graph is just a finite set of vertices with (undirected) edges connecting these vertices. For many different purposes there are however many different kinds of graphs, such as directed graphs, where the direction of the edges matters, and labelled graphs where either the vertices or the edges (or both) get assigned some labels to carry additional information.

To study the symmetries of a graph, one needs to consider all permutations of the vertices that leave the graph unchanged. In other words, when given a simple graph $\Gamma = (V, E)$ and a permutation $\sigma$ of $V$, then $\sigma$ is an automorphism of $\Gamma$ if and only if

$$\sigma(i) \sim \sigma(j) \iff i \sim j.$$

This relationship can also equivalently be expressed in the form of permutation matrices: if $A_\Gamma$ is the adjacency matrix of $\Gamma$ and $P_\sigma$ the permutation matrix belonging to $\sigma$ then $\sigma$ is an automorphism of $\Gamma$ if and only if

$$A_\Gamma P_\sigma = P_\sigma A_\Gamma.$$

For the automorphism group this means it can be expressed as

$$\mathrm{Aut}(\Gamma) = \{P_\sigma \in S_n \mid A_\Gamma P_\sigma = P_\sigma A_\Gamma\}.$$

There are many results on symmetries of graphs, such as Frucht's theorem [38], stating that any finite group can be realized as the automorphism group of a simple graph, or the result by Erdős and Rényi [34] that almost all graphs have trivial automorphism group.

In order to generalise the concept of the automorphism group of a graph to quantum groups, Bichon gave a definition of a quantum automorphism group in 2003 [13] and shortly afterwards, Banica gave a slightly different definition in [4]. In the literature it is now more common to study Banica's version, which we will also do in this thesis. This definition is based on the characterisation of the automorphism group of a graph via permutation matrices and is defined as follows, given a graph $\Gamma$ on $n$ vertices:

$$C(G^+_{aut}(\Gamma)) \coloneqq C^*(u_{ij} \mid u_{ij} = u_{ij}^2 = u_{ij}^*, \sum_{k=1}^n u_{ik} = 1 = \sum_{k=1}^n u_{ki}, A_\Gamma u = u A_\Gamma).$$

We will always have that $\mathrm{Aut}(\Gamma) \subseteq G^+_{aut}(\Gamma)$ as compact matrix quantum groups, and if $C(G^+_{aut}(\Gamma))$ is commutative, we have equality. The natural question to ask is therefore: for which graphs do the graph induced relations already yield commutativity? In the case that $C(G^+_{aut})$ is not commutative, and therefore $G^+_{aut}(\Gamma)$ is strictly larger than $\mathrm{Aut}(\Gamma)$, we say that $\Gamma$ has quantum symmetries.

This field of study has seen renewed progress recently, with many publications in the past few years. Previous work on this question includes a computer assisted approach by Eder, Levandovskyy, Schmidt, Steenpass, Weber and the author of this thesis in [48], where they compute the existence of quantum symmetries for all connected graphs on up to 6 vertices and on all graphs on 7 vertices with automorphism group of order less than or equal to 2. Another impressive result was found by van Dobben de Bruyn, Roberson, and Schmidt in [31] where they give a construction for asymmetric graphs that have quantum symmetries, answering the question for the existence of such a graph. Van Dobben de Bruyn, Nigam Kar, Roberson, Schmidt and Zeman also give a complete characterisation of quantum automorphism groups of trees in [30], including a polynomial-time algorithm to compute the quantum automorphism group of a given tree. In [42] Gromada investigates quantum automorphisms of Hadamard matrices and shows that these results transfer to the corresponding Hadamard graphs.

This thesis focuses on the study of quantum symmetries of a particular class of graphs, namely vertex-transitive graphs. Previous results on the study of this class include [6], where Banica and Bichon describe the quantum automorphism group for all vertex-transitive graphs on up to 11 vertices except for the Petersen graph. The case of the Petersen graph was later filled in by Schmidt in [74]. Moreover, in [22], [23] Chassaniol described the quantum automorphism group of all vertex-transitive graphs on 13 vertices. In this thesis, we fill the gap left by the previous results and compute the existence of quantum symmetries for all vertex-transitive graphs on 12 vertices.

**Matroids.** Matroids are combinatorial structures that were introduced by Whitney [88] in 1935. They are a generalisation of several mathematical concepts, among them both graphs and linear dependence. As such they form a bridge between these different aspects of mathematics and allow for a more general approach to some of these subjects. One example where the study of matroids is useful is in certain engineering problems, where the rank of systems of equations is of interest, for example in the work on rigidity of spatial structures. Such examples and more can be found in the book Matroid Applications [86].

The study of quantum symmetries of graphs has inspired a similar study for other objects. For example in [17], [24], the authors extend the study of quantum symmetries to quantum graphs. In [36], Faroß introduces the quantum automorphism groups of hypergraphs.

In this thesis, we extend the list above by proposing several definitions for quantum automorphism groups of matroids. There are several definitions for matroids that can be transferred into one another. When considering the automorphism group of a matroid in these different definitions, one can see that they are independent of the choice of definition. However, for the quantum automorphism groups it turns out that this is not the case and we get a chain of inclusion of the quantum automorphism groups.

### Quantum isomorphism of graphs

Related to the study of quantum automorphism groups of graphs is the study of quantum isomorphisms of graphs. They were introduced in [2] by Atserias, Mančinska, Roberson, Šámal, Severini and Varvitsiotis, where the authors designed a nonlocal game that captured classical isomorphism in its perfect classical strategies. They then defined two graphs to be quantum isomorphic if there exists a perfect quantum strategy for the graph isomorphism game for these graphs. Equivalently, two graphs are quantum isomorphic if a quantum permutation matrix exists that intertwines the adjacency matrices, i.e. the graph isomorphism game captures isomorphism in the following sense. If $\Gamma_1$ and $\Gamma_2$ are finite, simple graphs with adjacency matrices $A_1$ and $A_2$ respectively, then we have:

- There is a perfect classical strategy for the isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there exists a permutation matrix $P$ such that $A_1P = PA_2$.
- There is a perfect quantum strategy for the isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there exists a quantum permutation matrix $u$ such that $A_1u = uA_2$.

In [2] the authors not only introduce quantum isomorphisms of graphs but they also show that there indeed exist graphs that are not isomorphic but that are quantum isomorphic.

Further results on quantum isomorphisms of graphs include a Lovász type theorem by Mančinska and Roberson in [55] where they show that quantum isomorphism of graphs is equivalent to the equality of homomorphism counts from all planar graphs.

The relation to the quantum automorphisms of graphs is that the representations of the quantum automorphism group of a graph $\Gamma$ are exactly the perfect quantum strategies of the $(\Gamma, \Gamma)$-isomorphism game. Moreover, in [51] Lupini, Mančinska and Roberson show that two graphs $\Gamma_1$ and $\Gamma_2$ are quantum isomorphic if and only if there are vertices $v_1 \in V(\Gamma_1)$ and $v_2 \in \Gamma_2$ that are in the same orbit of the quantum automorphism group of the disjoint union of $\Gamma_1$ and $\Gamma_2$.

Inspired by the isomorphism game, we design a nonlocal game that captures switching isomorphism of signed graphs in a similar way. Signed graphs are graphs that have assigned to each edge a label of either $+1$ or $-1$. Two signed graphs are switching isomorphic if the delabelled versions are isomorphic such that additionally for any cycle the product of the labels of all edges appearing in the cycle is unchanged under the isomorphism. Equivalently, they are switching isomorphic if there exists a signed permutation matrix that intertwines the adjacency matrices. Our new nonlocal game now captures switching isomorphism in the following sense. If $\Gamma_1$ and $\Gamma_2$ are signed graphs with adjacency matrices $A_1$ and $A_2$, then we have:

- There is a perfect classical strategy for the switching isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there is a signed permutation matrix $H \in H_n$ such that $A_1 H = H A_2$. Here, $H_n$ denotes the hyperoctahedral group $H_n = S_2 \wr S_n$.
- There is a perfect quantum strategy for the switching isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there is a quantum signed permutation matrix $v$ such that $A_1 v = v A_2$.

In the second case we then call the two graphs quantum switching isomorphic.

### Outline and main results

In Chapter 1 we collect basic definitions and results around graphs, $C^*$-algebras, quantum groups and nonlocal games that are needed for the later chapters. We introduce the basic definitions and previous results around quantum automorphism groups of graphs in Chapter 2. We also collect some lemmas that are helpful when trying to compute the existence of quantum symmetries for a specific graph and give a general strategy that one can try to apply in such a case. In Chapter 3 we introduce basic notions around Gröbner bases and present a noncommutative version of the Buchberger algorithm, which has been implemented in `OSCAR` [66] by Schultz and the author.

We study the existence of quantum symmetries for all vertex-transitive graphs on 12 vertices in Chapter 4. We study these graphs in 5 different subclasses: disconnected graphs, products of smaller graphs, circulant graphs, semicirculant graphs and special cases that do not fit into any of the other subclasses. We find that for these graphs the existence of quantum symmetries is equivalent to the existence of disjoint automorphisms, which together with previous results on quantum symmetries of vertex-transitive graphs leads to the following theorem:

Theorem A (Theorem 4.1). *For vertex transitive graphs on up to* 13 *vertices, the existence of quantum symmetries is completely determined.*

Moreover, we compute the quantum automorphism group explicitly for two of these vertex-transitive graphs on 12 vertices for which this result does not follow directly from previous results. One of them is a circulant graph and the other a semi-circulant graph, both of which are graphs based on the cycle graph and adding some additional edges between vertices at certain distances.

Theorem B (Theorems 4.5.1 and 4.5.2). *The quantum automorphism group of* $C_{12}(4,5)$ *and* $C_{12}(3^+,6)$ *is* $H_2^+ \times S_3$.

We study quantum switching isomorphisms of signed graphs in Chapter 5. First, we give a definition of quantum switching isomorphism via the newly constructed switching isomorphism game and then characterise them in terms of compact matrix quantum groups.

THEOREM C (Theorem 5.2.7). *Two finite, simple signed graphs $\Gamma_1$ and $\Gamma_2$ with adjacency matrices $A_1$ and $A_2$ respectively are quantum switching isomorphic if and only if there exists a quantum signed permutation matrix $v$ with entries from a unital $C^*$-algebra admitting a faithful tracial state such that $vA_1 = A_2v$.*

We moreover find that for connected signed graphs any true quantum switching isomorphism always comes from a quantum isomorphism of the delabelled versions of the graphs.

THEOREM D (Theorem 5.3.2). *Let $\Gamma_1$ and $\Gamma_2$ be two connected signed graphs that are quantum switching isomorphic and let $v$ be a the quantum signed permutation matrix given by the quantum switching isomorphism. Then $v$ can be written as $v_{ij} = s_i u_{ij}$ for a quantum permutation matrix $u$ and some self-adjoint unitaries $s_i$ and we have that if $v$ has any noncommuting entries, then already $u$ must have noncommuting entries.*

In Chapter 6 we give definitions of different quantum automorphism groups of matroids according to different axiom systems of matroids, namely for flats, bases, independent sets and circuits and denote them by $G^{\mathcal{F}}_{aut}$, $G^{\mathcal{B}}_{aut}$, $G^{\mathcal{I}}_{aut}$ and $G^{\mathcal{C}}_{aut}$ respectively. For these quantum groups, we find the following inclusion result.

THEOREM E (Theorem 6.1). *For every matroid $\mathsf{M}$ we have*

$$\mathrm{Aut}(\mathsf{M}) = G^{\mathcal{F}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{B}}_{aut}(\mathsf{M}) = G^{\mathcal{I}}_{aut}(\mathsf{M}).$$

*If $\mathsf{M}$ is a simple rank $3$ matroid and the ground set $E(\mathsf{M})$ is not equal to $F_1 \cup F_2 \cup F_3$ for triangles $\{F_1, F_2, F_3\}$, then*

$$\mathrm{Aut}(\mathsf{M}) = G^{\mathcal{F}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{C}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{B}}_{aut}(\mathsf{M}) = G^{\mathcal{I}}_{aut}(\mathsf{M}).$$

We also find that for a significant class of matroids, the quantum automorphism group $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is already classical.

THEOREM F (Theorem 6.3.3). *If $\mathsf{M}$ is a matroid with $\mathrm{girth}(\mathsf{M}) \geq 4$, then*

$$G^{\mathcal{B}}_{aut}(\mathsf{M}) = G^{\mathcal{I}}_{aut}(\mathsf{M}) = \mathrm{Aut}(\mathsf{M}).$$

Moreover, we used Gröbner basis computations to check for a number of matroids whether $C(G^{\mathcal{B}}_{aut}(\mathsf{M}))$ and $C(G^{\mathcal{C}}_{aut}(\mathsf{M}))$ are commutative and collect the results in Section 6.5.

Lastly we collect some open questions related to the subjects covered in this thesis in Chapter 7.

CHAPTER 1

# Preliminaries

## 1.1. Graphs

Large parts of this thesis are about graphs. We therefore start by collecting some basic definitions and facts about graphs.

1.1.1. Definition. A *finite graph without multiple edges* $\Gamma$ consists of a finite vertex set $V(\Gamma)$ and an edge set $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. It is called *undirected*, if for any edge $(u, v) \in E(\Gamma)$ it also holds that $(v, u) \in E(\Gamma)$. A *loop* is an edge of the form $(i, i) \in E(\Gamma)$.

The *adjacency matrix* $A_\Gamma$ of a graph on $n$ vertices is the $n \times n$-matrix with the $(i, j)$-entry being the number of edges from vertex $i$ to vertex $j$. If $\Gamma$ is undirected and without multiple edges, $A_\Gamma$ is thus a symmetric matrix with $\{0, 1\}$-entries.

In an undirected graph, we will write $u \sim v$ to mean that $(u, v) \in E(\Gamma)$ and thus also $(v, u) \in E(\Gamma)$.

**All graphs considered in this work will be undirected and without multiple edges or loops.**

1.1.2. Definition. Two (undirected) graphs $\Gamma_1$ and $\Gamma_2$ are *isomorphic* if there exists a bijection $\sigma : V(\Gamma_1) \to V(\Gamma_2)$ such that $\sigma$ preserves adjacency, i.e.

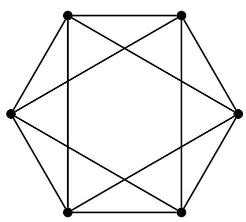
$$i \sim j \iff \sigma(i) \sim \sigma(j) \text{ for all } i, j \in V(\Gamma_1).$$

1.1.3. Definition. Given a graph $\Gamma$ on $n$ vertices with adjacency matrix $A_\Gamma$, its automorphism group $G_{aut}(\Gamma)$ is the group of all automorphisms of $\Gamma$ and can be expressed as a group of permutation matrices in the following sense:

$$G_{aut}(\Gamma) = \{P \in S_n | PA_\Gamma = A_\Gamma P\}.$$

1.1.4. Definition. A graph $\Gamma$ is *vertex-transitive* if for any two vertices $i, j \in V(\Gamma)$ there exists an automorphism $\varphi \in G_{aut}(\Gamma)$ such that $\varphi(i) = j$.

1.1.5. Example. The following graph is vertex-transitive:



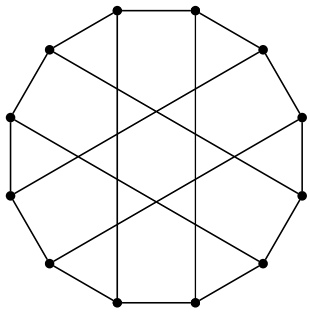
It is called $C_6(2)$ and is an example from the family of so-called *circulant graphs*.

1.1.6. Definition. For $1 < k_1 < \cdots < k_r \leq \left\lfloor \frac{n}{2} \right\rfloor$, we define the the *circulant graph* on $n$ vertices with chords in distances $k_1, \ldots, k_r$, as the graph obtained by drawing the $n-$cycle $C_n$ and then connecting all pairs of vertices in distance $k_i$, for any $i$. We write $C_n(k_1, \ldots, k_r)$ for this graph. In other words, the vertex set of $C_n(k_1, \ldots, k_r)$ is $V = \{1, \ldots, n\}$ and for $i, j \in V$ we have

$$i \sim j \Leftrightarrow |i - j| \mod n \in \{1, k_1, \ldots, k_r\}.$$

Related to this is the family of *semi-circulant graphs*: the semi-circulant graph $C_n(k_1, \ldots, k_r, l_1^+, \ldots, l_s^+)$ is constructed by taking the circulant graph $C_n(k_1, \ldots, k_r)$ and adding edges between $i, j \in V$ if $i$ is even, $i < j$ and $j - i \in \{l_1^+, \ldots, l_s^+\}$.

1.1.7. Example. The graph $C_{12}(5^+)$ looks as follows:

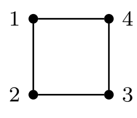


1.1.8. Definition. Given a graph $\Gamma$ and two automorphisms $\varphi$ and $\psi$ of $\Gamma$, we say that $\varphi$ and $\psi$ are *disjoint* if we have

(i) $\varphi(v) \neq v \implies \psi(v) = v$ and
(ii) $\psi(v) \neq v \implies \varphi(v) = v$

for all vertices $v \in V(\Gamma)$. We say that $\Gamma$ *has disjoint automorphisms*, if there are two non-trivial disjoint automorphisms in $G_{aut}(\Gamma)$.

1.1.9. Example. The graph $C_4$, given as follows, has the disjoint automorphisms $\varphi = (1\,3)$ and $\psi = (2\,4)$.




## 1.2. $C^*$-Algebras

We will now recall the basic definitions and some important statements about $C^*$-algebras. Most of the definitions and statements in this section come from the lecture notes [54], [84]. More detailled information about $C^*$-algebras can for example be found in the books [15], [64].

1.2.1. Definition. A *$C^*$-algebra* $\mathcal{A}$ is a Banach algebra which has an antilinear map $^* : \mathcal{A} \to \mathcal{A}$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$, called the involution, whose norm satisfies the $C^*$-identity:

$$\|x^*x\|^2 = \|x\|^2 \text{ for all } x \in \mathcal{A}.$$

1.2.2. Example. Given a Hilbert space $\mathcal{H}$, the algebra of all bounded operators on $\mathcal{H}$, $B(\mathcal{H})$, is a $C^*$-algebra with the adjoint map being the involution.

Another example is given by the continuous functions $C(X)$ on a compact space $X$. In fact, one can see that all commutative $C^*$-algebras arise in such a way, as can be seen in the following theorem.

The following theorem is an important fact in $C^*$-algebra theory and opens the interpretation of noncommutative $C^*$-algebras as "noncommutative" compact spaces.

1.2.3. Theorem (Gelfand-Naimark).

*(i) If $X$ is a compact topological space, then $C(X)$ is a commutative unital $C^*$-algebra.*

*(ii) If $A$ is a commutative $C^*$-algebra, then there exists a compact topological space $X$ such that $A \cong C(X)$. In fact one can see that $X = \mathrm{Spec}(A)$, where $\mathrm{Spec}(A)$ is the spectrum of $A$, i.e. the set of all characters of $A$.*

An important tool in the study of $C^*$-algebras are universal constructions. In fact all $C^*$-algebras considered in this thesis will be costructed in such a way:

1.2.4. Definition. Let $E = \{x_i \mid i \in I\}$ for any index set $I$ be a set of generators and let the set $E^* := \{x_i^* \mid i \in I\}$ be disjoint from $E$. We define the *free $^*$-algebra* $P(E)$ on the generator set as the algebra of all polynomials in $E$ and $E^*$ with straightforward addition, multiplication and involution and with coefficients in $\mathbb{C}$.

Let now $R \subseteq P(E)$ be a set of polynomials in $E$ and $E^*$ and define $J(R)$ as the two-sided $^*$-ideal generated by $R$. We say

$$A(E, R) := P(E)/J(R)$$

is the *universal involutive algebra with generators $E$ and relations $R$*. For $x \in A(E, R)$, we put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E, R)\}.$$

If now $\|x\| < \infty$ for all $x \in A(E, R)$ we define

$$C^*(E \mid R) := \overline{A(E, R)/\{x \in A(E, R) \mid \|x\| = 0\}}^{\|\cdot\|}$$

as the *universal $C^*$-algebra with generators $E$ and relations $R$* and we say that the universal $C^*$-algebra $C^*(E \mid R)$ *exists*.

The universal $C^*$-algebra $C^*(E \mid R)$ has the following universal property:

1.2.5. PROPOSITION. *If $B$ is a $C^*$-algebra containing a set $E' := \{x_i' \mid i \in I\}$ such that $E'$ satisfies the relations $R$, then there exists a unique ${}^*$-homomorphism $\varphi : C^*(E \mid R) \to B$ with $\varphi(x_i) = x_i'$.*

1.2.6. EXAMPLE.

(i) The universal $C^*$-algebra generated by a single self-adjoint element does not exist, since there exist self-adjoint elements with arbitrarily large norm and therefore the map

$$\begin{aligned} \pi_y : A(x, x = x^*) &\to \mathbb{C} \\ x &\to y \end{aligned}$$

for $y = \overline{y}$ yields a family of seminorms

$$p_y(z) := |\pi_y(z)|$$

such that

$$\|x\| = \sup_{y=\overline{y}\in\mathbb{C}} \{p_y(x)\} = \infty$$

for the generator $x \in A(x, x = x^*)$.

(ii) For any $n \geq 2$ the universal $C^*$-algebra

$$C^*(e_{ij}, 1 \leq i, j \leq n \mid e_{ij}^* = e_{ji},\ e_{ij}e_{kl} = \delta_{jk}e_{il})$$

is isomorphic to $M_n(\mathbb{C})$. A set of nonzero elements $\{f_{ij}\}$ in a $C^*$-algebra $A$ satisfying the above relations is called a set of *matrix units* of type $M_n(\mathbb{C})$ in $A$.

Taking the universal $C^*$-algebra generated by an infinite set of matrix units one gets the compact operators on a separable Hilbert space $H$:

$$C^*(e_{ij}, i, j \in \mathbb{N} \mid e_{ij}^* = e_{ij},\ e_{ij}e_{kl} = \delta_{jk}e_{il}) \cong K(H).$$

(iii) Let $\mathbb{S}^1$ be the unit circle. Then

$$C(\mathbb{S}^1) \cong C^*(u, 1 \mid u^*u = uu^* = 1),$$

i.e. the continuous functions on the unit circle are just the universal $C^*$-algebra generated by a unitary. This universal $C^*$-algebra does exist, since we have $p(u) \leq 1$ for any $C^*$-seminorm $p$ and a unitary element $u$ and since it suffices to check the finiteness of the norm on generators.

Whenever we add a 1 to the generators, we implicitly also add the relations $1 = 1^2 = 1^*$ and the relations that 1 actually is a unit with respect to multiplication.

(iv) There is also a universal $C^*$-algebra generated by a single isometry, called the *Toeplitz algebra*.

$$\mathcal{T} = C^*(v, 1 \mid v^*v = 1).$$

One can see that the ideal generated by $1 - vv^*$ is isomorphic to the compact operators $K(H)$ on a seperable Hilbert space $H$ by checking that the elements $e_{ij} = v^i(1 - vv^*)v^{*j}$ satisfy the relations mentioned in (ii). Since we also have $\mathcal{T}/\langle 1 - vv^* \rangle \cong C(\mathbb{S}^1)$ we get that the following sequence is exact:

$$0 \to K(H) \to \mathcal{T} \to C(\mathbb{S}^1) \to 0.$$

(v) For a discrete group $G$ the maximal group $C^*$-algebra $C^*(G)$ can be expressed as a universal $C^*$-algebra:

$$C^*(G) = C^*(u_g, g \in G \mid u_g \text{ unitary}, u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}),$$

where $u_e = 1$ for the neutral element $e \in G$. We have for example that the group $C^*$-algebra of $\mathbb{Z}$ is just generated by a single unitary, i.e. we have $C^*(\mathbb{Z}) \cong C(\mathbb{S}^1)$ and similarly we get $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$.

The group $C^*$-algebra of the free group on $n$ generators $\mathbb{F}_n$ is another important example, which has the group $C^*$-algebra $C^*(\mathbb{F}_n)$ which is generated by $n$ unitaries.

1.2.7. Remark. In the case that all relations in $R$ from Definition 1.2.4 only have integer coefficients it is also possible to consider the free $*$-algebra $P_{\mathbb{Z}}(E)$ with integer coefficients, since then the quotient $P_{\mathbb{Z}}(E)/J(R)$ is defined. Doing the rest of the construction in the same way as above yields an *integral universal $C^*$-algebra* $C^*_{\mathbb{Z}}(E \mid R)$. Viewing the additive group of this algebra as a $\mathbb{Z}$-module allows tensoring over $\mathbb{Z}$, from which it follows that $C^*_{\mathbb{Z}}(E \mid R) \otimes \mathbb{C}$ is isomorphic as a $*$-algebra to $C^*(E \mid R)$. This is useful when doing computations on a computer, which we do later, where we do computations over $C^*_{\mathbb{Z}}(E \mid R) \otimes \mathbb{Q}$.

An important fact about $C^*$-algebras is the so-called *GNS-construction*, which provides for any $C^*$-algebra a concrete Hilbert space and a faithful representation on this Hilbert space.

1.2.8. Theorem (GNS Construction). *Let $A$ be a $C^*$-algebra and $\varphi$ be a state on $A$. Putting*

$$\langle x, y \rangle_\varphi := \varphi(y^* x) \text{ and } \mathcal{N}_\varphi := \{x \in A \mid \langle x, x \rangle_\varphi = 0\}$$

*we obtain an inner product on $K_\varphi := A/\mathcal{N}_\varphi$ making it into a pre-Hilbert space.*

*Defining $\pi^0_\varphi(a)$ for $a \in A$ as the left-multiplication operator by $a$ on $A/\mathcal{N}_\varphi$, i.e. putting*

$$\pi^0_\varphi(a)(x + \mathcal{N}_\varphi) := ax + \mathcal{N}_\varphi$$

*we obtain a bounded operator $\pi^0_\varphi(a)$ with $\|\pi^0_\varphi(a)\| \leq \|a\|$ that extends to a bounded operator $\pi_\varphi(a)$ on the closure $H_\varphi := \overline{K_\varphi}^{\|\cdot\|_\varphi}$. This $\pi_\varphi$ is called the* GNS representation *of $A$ associated to $\varphi$.*

*There also exists a unique cyclic vector* $\xi_\varphi$ *making the representation cyclic such that* $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$. *In the case where* $A$ *is unital, this* $\xi_\varphi$ *is just the image of* $1_A$ *in* $H_\varphi$. *Otherwise, one can take the unique extension* $\tilde{\varphi}$ *of* $\varphi$ *to the unitisation* $\tilde{A}$ *with* $\|\tilde{\varphi}\| = \|\varphi\|$. *Then one can identify* $H_\varphi$ *with* $H_{\tilde{\varphi}}$ *and one can take* $\xi_\varphi$ *to be* $\xi_{\tilde{\varphi}}$.

From the above construction another important fact about $C^*$-algebras follows.

1.2.9. Corollary. *Every* $C^*$*-algebra* $A$ *admits a faithful representation* $\pi : A \hookrightarrow B(H)$. *Therefore,* $A$ *is isomorphic to a* $C^*$*-subalgebra of* $B(H)$.

When considering tensor products of $C^*$-algebras, there are a number of technicalities that have to be kept in mind. When given two $C^*$-algebras, it is always possible to take the algebraic tensor product of the two algebras. The question about the $C^*$-norm however is not quite so straightforward.

1.2.10. Definition. Let $A$ and $B$ be two $C^*$-algebras and denote by $A \odot B$ their algebraic tensor product. There are several ways to define a $C^*$-norm on this tensor product, of which we will mention the most important two.

For any $\sum_{i=1}^n a_i \otimes b_i \in A \odot B$, we put

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{\max} := \sup \left\{ \Big\| \pi(\sum_{i=1}^n a_i \otimes b_i) \Big\| \mid \pi : A \odot B \to B(H) \text{ }^*\text{-homomorphism} \right\}.$$

We then call $A \otimes_{\max} B := \overline{A \odot B}^{\|\cdot\|_{\max}}$ the *maximal tensor product* of $A$ and $B$. The norm $\|\cdot\|_{\max}$ is, as the name suggests, the largest possible $C^*$-norm on $A \odot B$.

For the *minimal tensor product* $A \otimes_{\min} B$ we put

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{\min} := \sup \left\{ \Big\| (\pi \otimes \sigma)(\sum_{i=1}^n a_i \otimes b_i) \Big\| \right\},$$

where the supremum is over all representations $\pi : A \to B(H)$ and $\sigma : B \to B(K)$ of $A$ and $B$ respectively. We then put $A \otimes_{\min} B := \overline{A \odot B}^{\|\cdot\|_{\min}}$. As above with the maximal norm, we also get that $\|\cdot\|_{\min}$ is the smallest possible $C^*$-norm on $A \odot B$.

For the rest of this thesis, we will denote the minimal tensor product $\otimes_{\min}$ by the symbol $\otimes$.

1.2.11. Definition. A $C^*$-algebra $A$ is called *nuclear* if for any $C^*$-algebra $B$ we have

$$A \otimes_{\min} B = A \otimes_{\max} B.$$

As a consequence of the minimality and maximality of the corresponding norms, we get that there is only one possible $C^*$-norm on $A \odot B$ and thus also only one completion and we write

$$A \otimes B := A \otimes_{\min} B = A \otimes_{\max} B.$$

1.2.12. Example.

- Any commutative $C^*$-algebra $A$ is nuclear.
- Finite dimensional $C^*$-algebras are nuclear.
- The group $C^*$-algebra of the free group $\mathbb{F}_2$ is not nuclear.

1.2.13. Proposition. *Let $A$ and $B$ be unital $C^*$-algebras. Then their maximal tensor product can be expressed as a universal $C^*$-algebra as follows:*

$$A \otimes_{\max} B = C^*(a \in A,\ \textit{relations of } A, b \in B,\ \textit{relations of } B \mid ab = ba, 1_A = 1_B).$$

Another form of products of $C^*$-algebras are free products.

1.2.14. Definition. Let $A$ and $B$ be $C^*$-algebras.

- The $C^*$-algebra
$$A * B \coloneqq C^*(a \in A, b \in B \mid \text{relations of } A,\ \text{relations of } B)$$
is called the *free product* of $A$ and $B$.
- If $A$ and $B$ are unital, one can form their *unital free product*
$$A *_{\mathbb{C}} B \coloneqq C^*(a \in A, b \in B \mid \text{relations of } A,\ \text{relations of } B, 1_A = 1_B).$$
- Let $C$ be a $C^*$-algebra and $j_1 : C \hookrightarrow A$, $j_2 : C \hookrightarrow B$ be two embeddings. Then the $C^*$-algebra
$$A *_C B \coloneqq C^*(a \in A, b \in B \mid \text{relations of } A,\ \text{relations of } B, j_1(x) = j_2(x) \text{ for all } x \in C)$$
is called the *amalgamated free product* of $A$ and $B$.

## 1.3. Compact Matrix Quantum Groups

In this section we collect basic facts about compact matrix quantum groups as defined by Woronowicz [89], [91]. More on compact quantum groups can be found in the books [65], [78]. We additionally use the expository paper [53], the PhD thesis [52] and the lecture notes [83] as sources.

1.3.1. Definition. A *compact matrix quantum group* (CMQG) $G$ is a unital $C^*$-algebra $C(G)$ equipped with a $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$ and a unitary $u \in M_n(C(G))$, $n \in \mathbb{N}$, such that

(i) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all $i, j$

(ii) $\bar{u}$ is an invertible matrix

(iii) the elements $u_{ij}$ $(1 \leq i, j \leq n)$ generate $C(G)$ (as a $C^*$-algebra).

The unitary $u$ is called the *fundamental corepresentation (matrix)* of $(C(G), \Delta, u)$. Since (i) and (iii) uniquely determine $\Delta$, one can also refer to the pair $(C(G), u)$ as a compact matrix quantum group.

If $G = (C(G), u)$ and $H = (C(H), v)$ are compact matrix quantum groups with $u \in M_n(C(G))$ and $v \in M_n(C(H))$, we say that $G$ is a *compact matrix quantum subgroup* of $H$, if there is a surjective $^*$-homomorphism from $C(H)$ to $C(G)$ mapping generators to generators. We then write $G \subseteq H$. If we have $G \subseteq H$ and $H \subseteq G$, they are said to be equal as compact matrix quantum groups.

More generally, we say $G$ is a *compact quantum subgroup* of $H$ if there is a surjective $*$-homomorphism $\varphi$ from $C(H)$ to $C(G)$, no longer necessarily sending generators to generators, satisfying

$$\Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta.$$

The name compact matrix quantum group is justified since the commutative CMQGs correspond exactly to the compact matrix groups.

1.3.2. Proposition. *Let $G = (C(G), \Delta, u)$ be a compact matrix quantum group where $u \in M_n(C(G))$ for an $n \in \mathbb{N}$ and assume that $C(G)$ is commutative. Then there exist a closed compact subgroup $G' \subseteq U_n(\mathbb{C})$ and an isomorphism $\Phi : C(G) \to C(G')$ such that $\Phi(u_{ij})(x) = x_{ij}$ for all $x \in G'$ and all $i, j = 1, \ldots, n$.*

For a proof of the above proposition see e.g. Proposition 6.1.11 in [78]. The proof uses the Gelfand-Naimark Theorem 1.2.3 and the fact that the compact space $X$ from Theorem 1.2.3 inherits a group structure from the comultiplication $\Delta$.

1.3.3. Example. The *quantum symmetric group* $S_n^+ = (C(S_n^+), u)$, which was first defined by Wang [82] in 1998, is the compact matrix quantum group given by

$$C\left(S_n^+\right) := C^*\left(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2,\ \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{ki} = 1\ \forall i, j = 1, \cdots, n\right)$$

and the $*$-homomorphism $\Delta$ is given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

It can be shown that the quotient of $C\left(S_n^+\right)$ by the relation that all $u_{ij}$ commute is exactly $C\left(S_n\right)$. Moreover, $S_n$ can be seen as a compact matrix quantum group $S_n = \left(C\left(S_n\right), u\right)$, where $u_{ij} : S_n \to \mathbb{C}$ are the evaluation maps of the matrix entries. We then have $S_n \subseteq S_n^+$ as compact matrix quantum groups, which justifies the name "quantum symmetric group".

1.3.4. Remark. For $n \in \{1, 2, 3\}$, one can see that the relations of $S_n^+$ already imply the commutation of all the generators, which means that for $n < 4$, we have $S_n = S_n^+$ as compact matrix quantum groups. For $n = 1$ it is obvious since there is only one generator, while for $n = 2$, the generator matrix is of the form

$$\begin{pmatrix} u_{11} & 1 - u_{11} \\ 1 - u_{11} & u_{11} \end{pmatrix}$$

and it is also immediate to see the commutativity.

For $n = 3$, only the commutativity of any generators in the same row or column are immediate, for the rest of the generator pairs, one will have to do some computations. We give the computation for $u_{11}$ and $u_{22}$ here, the rest of the computations are similar.

$$\begin{aligned}
u_{11}u_{22} &= (1 - u_{12} - u_{13})u_{22}(u_{11} + u_{21} + u_{31}) \\
&= u_{22}u_{11} + u_{22}u_{31} - u_{13}u_{22}u_{11} - u_{13}u_{22}u_{31} \\
&= u_{22}u_{11} + u_{22}u_{31} - u_{13}(1 - u_{21} - u_{23})u_{11} - u_{13}(1 - u_{12} - u_{32})u_{31} \\
&= u_{22}u_{11} + u_{22}u_{31} - u_{13}u_{31} \\
&= u_{22}u_{11} + (1 - u_{21} - u_{23})u_{31} - (1 - u_{23} - u_{33})u_{31} \\
&= u_{22}u_{11} + u_{31} - u_{23}u_{31} - u_{31} + u_{23}u_{31} \\
&= u_{22}u_{11}.
\end{aligned}$$

Here we repeatedly use the fact that the sum over rows or columns is equal to 1 and the orthogonality of generators that are in the same row or column.

For $n \geq 4$ however, one can see that $S_n^+$ is not commutative: let

$$A \coloneqq C^*(1, p, q \mid p, q \text{ projections })$$

be the universal $C^*$-algebra generated by two projections. Mapping the generator matrix $u$ of $S_4^+$ to

$$\tilde{u} \coloneqq \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

we get a surjective $*$-homomorphism from $C(S_4^+)$ to $A$ by the universal property of $C(S_4^+)$ which has a noncommutative image, and therefore $C(S_4^+)$ is also noncommutative. For $n > 4$ one can construct a similar $*$-homomorphism in the same way by mapping to a matrix with $\tilde{u}$ in the upper left corner and the identity on the rest of the matrix to match the dimensions.

1.3.5. Definition. We call quantum subgroups of the quantum symmetric group $S_n^+$ *quantum permutation groups.*

1.3.6. Definition. We call any matrix that satisfies the relations of $S_n^+$ a quantum permutation matrix. In other words, a matrix $u$ is a *quantum permutation matrix* if all of its entries are orthogonal projections and the rows and columns of $u$ each sum up to 1.

In a similar way that $S_n^+$ is a quantisation of $S_n$, it is possible to quantise the orthogonal group $O_n$ and the unitary group $U_n$.

1.3.7. Example.

- One can view the orthogonal group $O_n$ as a compact matrix quantum group defined via the universal $C^*$-algebra

$$C(O_n) = C^*(u_{ij},\, 1 \leq i, j \leq n \mid u_{ij} = u_{ij}^*, u = (u_{ij}) \text{ orthogonal, all } u_{ij} \text{ commute}).$$

  In the 1990's, Wang [80] defined the free orthogonal quantum group $O_n^+$ by

$$C(O_n^+) = C^*(u_{ij},\, 1 \leq i, j \leq n \mid u_{ij} = u_{ij}^*, u = (u_{ij}) \text{ orthogonal}).$$

  The fact that $u$ is orthogonal is equivalent to the relations

$$\sum_k u_{ik} u_{jk} = \sum_k u_{ki} u_{kj} = \delta_{ij} \text{ for all } i, j.$$

  Checking the relations it is easy to see that both $O_n$ and $S_n^+$, and therefore also $S_n$, are compact matrix quantum subgroups of $O_n^+$.
- In the same article as above [80], Wang also introduced the free unitary quantum group $U_n^+$ defined by the following universal $C^*$-algebra:

$$C(U_n^+) := C^*(u_{ij}, 1 \leq i, j \leq n \mid u, u^t \text{ unitary}).$$

  Equivalently, one can take the $C^*$-algebra

$$C(U_n^+) = C^*(u_{ij} \mid u, \overline{u} \text{ unitary}).$$

  The unitary condition on $u$ is equivalent to the relations

$$\sum_k u_{ik} u_{jk}^* = \sum_k u_{ki}^* u_{kj} = \delta_{ij} \text{ for all } i, j$$

  while the unitary condition on $\overline{u}$ gives the relations

$$\sum_k u_{ik}^* u_{jk} = \sum_k u_{ki} u_{kj}^* = \delta_{ij} \text{ for all } i, j.$$

  Unlike in the case where all $u_{ij}$ commute, the condition that $\overline{u}$ is unitary is not implied by the condition that $u$ is unitary. In fact, the $C^*$-algebra

$$C^*(u_{ij} \mid u \text{ unitary})$$

  does not define a CMQG, since $\overline{u}$ is not invertible, see Example 4.1 in [80].

  One can see in the relations of $U_n^+$ that $O_n^+$ is just the quotient of $U_n^+$ by the relation $u = \overline{u}$ and we thus have $O_n^+ \subseteq U_n^+$.

In [14], Bichon defined the quantum analogue of the wreath product in order to be able to describe the quantum automorphism group of identical copies of a graph. It allows to construct new quantum groups out of any compact quantum group and a quantum permutation group.

1.3.8. Definition. Let $G = (C(G), u)$ be a compact matrix quantum group and let $H = (C(H), v)$ be a quantum permutation group with $u \in M_n(C(G))$ and $v \in M_m(C(H))$.

The free wreath product $G \wr_* H = (C(G) *_w C(H), w)$ is a CMQG where the fundamental corepresentation $w$ is defined by $(w_{ia,jb}) = (u_{ij}^{(a)} v_{ab}) \in M_{nm}(C(G) *_w C(H))$. Here $u^{(a)}$ are copies of $u$ and $C(G) *_w C(H)$ is the universal $C^*$-algebra generated by $u_{ij}^{(a)}$, $v_{ab}$ with the relations of the free product $C(G) * C(H)$ and additionally the relations $u_{ij}^{(a)} v_{ab} = v_{ab} u_{ij}^{(a)}$.

1.3.9. Example. The hyperoctahedral quantum group $H_n^+$ is a quantisation of the hyperoctahedral group $H_n$ and was first defined by Bichon in [14] and further studied by Banica, Bichon and Collins in [8]. It is defined as the free wreath product of $S_2$ with $S_n^+$ :

$$C(H_n^+) \coloneqq C(S_2) \wr_* C(S_n^+).$$

This is analogue to the classical hyperoctahedral group, which can be written as

$$H_n = \mathbb{Z}_2 \wr S_n = S_2 \wr S_n.$$

Writing the hyperoctahedral quantum group as a universal $C^*$-algebra is possible as follows:

$$C(H_n^+) = (v_{ij} \mid v_{ij}^* = v_{ij}, v_{ij}^3 = v_{ij}, \sum_{i=1}^n v_{ik}^2 = \sum_{j=1}^n v_{kj}^2 = 1).$$

More generally, one can even consider the quantum group $H_n^{s+}$, which was introduced in [3] as the quantum version of $H_n^s$ and studied further in [11] where the authors showed that

$$H_n^{s+} = \mathbb{Z}_s \wr_* S_n^+.$$

The above is equivalent to the following definition as a universal $C^*$-algebra:

$$\begin{aligned} C(H_n^+) = C^*(u_{ij},\, 1 \leq i,j \leq n \mid\, & u \text{ unitary}, u^t \text{ unitary}, \\ & p_{ij} = u_{ij} u_{ij}^* \text{ is a projection}, u_{ij}^s = p_{ij}). \end{aligned}$$

These quantum groups $H_n^{s+}$ have some of the earlier quantum groups as special cases: for $s = 1$, we get that $H_n^{s+} = S_n^+$ and for $s = 2$ we get $H_n^{s+} = H_n^+$.

1.3.10. Example. In [11] it has been shown that for any $n$ and any $s$, the compact quantum group $H_n^{s+}$ is a quantum permutation group, i.e. there exists some $N \in \mathbb{N}$ such that $H_n^{s+}$ is a compact quantum subgroup of $S_N^+$. To illustrate this, we will show here that $H_2^+$ is a quantum subgroup of $S_4^+$.

Let $v$ be the fundamental corepresentation matrix of $H_2^+$. Then the elements

$$p_{ij} \coloneqq \frac{v_{ij}^2 + v_{ij}}{2} \text{ and } q_{ij} \coloneqq \frac{v_{ij}^2 - v_{ij}}{2}$$

are projections that satisfy

$$v_{ij} = p_{ij} - q_{ij}$$

and therefore also

$$v_{ij}^2 = p_{ij} + q_{ij}.$$

We now put

$$u' := \begin{pmatrix} p_{11} & q_{11} & p_{12} & q_{12} \\ q_{11} & p_{11} & q_{12} & p_{12} \\ p_{21} & q_{21} & p_{22} & q_{22} \\ q_{21} & p_{21} & q_{22} & p_{22} \end{pmatrix}.$$

Then $u'$ fulfills the relations of $C(S_4^+)$ since all entries are projections and since $p_{ij} + q_{ij} = v_{ij}^2$ and we get the relations that rows and columns sum to one from the corresponding relation from $H_2^+$. The universal property of $S_4^+$ thus yields a homomorphism mapping the fundamental representation matrix of $S_4^+$ to $u'$, and since we can construct the $v_{ij}$ from the $p_{ij}$ and $q_{ij}$ it is also surjective.

1.3.11. Remark. The theory of compact matrix quantum groups is embedded in the larger theory of compact quantum groups, which was introduced later by Woronowicz in [92] in the sense that every CMQG is also a compact quantum group.

A reason to study compact quantum groups is that while the dual of an abelian group is always again a group, the same cannot be said for general groups, however the dual of any group will always be a quantum group.

An important fact about compact quantum groups, and therefore also about compact matrix quantum groups, is the existence of a Haar state, which corresponds to the Haar measure of compact groups.

1.3.12. Theorem. *Let $G = (C(G), \Delta, u)$ be a CMQG. Then there exists a unique state $h$, called the* Haar state *of $G$*

$$h : C(G) \to \mathbb{C}$$

*satisfying*

$$(\mathrm{id} \otimes h) \circ \Delta = (h \otimes \mathrm{id}) \circ \Delta = h \cdot 1_{C(G)}.$$

1.3.13. Remark. In [51], the authors have computed the value of the Haar state of quantum permutation groups on products of pairs of operators using only the size of the orbitals of the quantum permutation group in question. Moreover, in [9], a formula for the value of the Haar state on arbitrary monomials in $O_n^+$ and $U_n^+$ was proven, called the Weingarten calculus. This result was later extended to monomials in any easy quantum group in [10].

Lastly, we want to introduce the representation theory of compact matrix quantum groups. It is a known fact from group theory, known as Tannaka-Krein duality, that any compact group can be recovered from its representation category [32]. A similar statement was shown by Woronowicz for compact matrix quantum groups [90] and later also in general for compact quantum groups by Wang [81].

1.3.14. Definition. Let $A$ be a unital $C^*$-algebra with a unital $*$-homomorphism $\Delta : A \to A \otimes A$. A *finite-dimensional representation* of $(A, \Delta)$ is a matrix

$$v \in M_m(A),\ m \in \mathbb{N} \text{ with } \Delta(v_{ij}) = \sum_{k=1}^{m} v_{ik} \otimes v_{kj}.$$

We say that $v$ is *non-degenerate* if it is invertible and if it is unitary, then we call $v$ a *unitary representation*. Denote by $H_v$ the Hilbert space belonging to $v$ such that we have $v \in B(H_v) \otimes A$.

If $G = (A, \Delta)$ is a CMQG, then the above defines finite dimensional representations of CMQGs.

1.3.15. Example. For any compact matrix quantum group $G = (C(G), u)$, the fundamental corepresentation matrix $u$ and the trivial representation $1 \in C(G)$ are always representations in the sense of the above definition.

In order to construct the category of representations that we want to use later, we will next define the morphisms of representations, which are called intertwiners.

1.3.16. Definition. Let $G = (C(G), u)$ be a CMQG and let $v \in B(H_v) \otimes C(G)$ and $v' \in B(H_{v'}) \otimes C(G)$ be two finite-dimensional representations with $\dim(H_v) = m$ and $\dim(H_{v'}) = m'$. Then a morphism between $v$ and $v'$, called an *intertwiner*, is given by a linear map $T \in B(H_v, H_{v'})$ such that

$$Tv = v'T.$$

We view $T$ as a matrix in $B(H_v, H_{v'}) \otimes \mathbb{C} \subseteq B(H_v, H_{v'}) \otimes C(G)$ and $Tv$ and $v'T$ are the matrix products of the respective matrices. The representations $v$ and $v'$ are called *equivalent* if $m = m'$ and if there exists an invertible intertwiner between $v$ and $v'$. Moreover, we call the representation $v$ *irreducible*, if every intertwiner $Tv = vT$ is of the form $T = \lambda \cdot \mathrm{id}$.

We next want to recall the decomposition properties of representations of compact matrix quantum groups.

1.3.17. Definition. Let $G = (C(G), \Delta, u)$ be a CMQG and let $v \in M_m(C(G))$ and $v' \in M_{m'}(C(G))$ be two finite dimensional representations of $G$.

- The *direct sum* of $v$ and $v'$ is given by the matrix $v \oplus v' \in M_{m+m'}(C(G))$.
- The *tensor product* of $v$ and $v'$ is given by the Kronecker product $v \otimes v' \in M_{mm'}(C(G))$ of $v$ and $v'$.

1.3.18. Theorem ([65]). *Let $G = (C(G), \Delta, u)$ be a CMQG.*

*(i) Every non-degenerate finite-dimensional representation of $G$ is equivalent to a unitary representation.*

*(ii) Every unitary representation of $G$ decomposes into a direct sum of irreducible finite-dimensional representations.*

We are now ready to consider the category of representations of a CMQG.

1.3.19. PROPOSITION ([90]). *Let $G = (C(G), \Delta, u)$ be a CMQG. We consider the category of representations of $G$ with the objects $R$ being all finite-dimensional unitary representations of $G$ and the morphisms given by the set of intertwiners*

$$\mathrm{Hom}(v, v') \coloneqq \{T \in B(H_v, H_{v'}) \mid Tv = v'T\}.$$

*Taking the tensor product $\otimes$ of representations and taking $H_v$ as the finite dimensional Hilbert space such that $v \in B(H_v) \otimes C(G)$ for $v \in R$ we get that the category defined as $(R, \otimes, \{H_v\}_{v\in R}, \{\mathrm{Hom}(v, v')\}_{v,v'\in R})$ is a concrete monoidal $W^*$-category in the sense that we have:*

- *(i) The identity operator $\mathrm{id}_v = \mathrm{id}_{H_v}$ belongs to $\mathrm{Hom}(v, v)$.*
- *(ii) $\mathrm{Hom}(v, v')$ is a linear subspace of $B(H_v, H_{v'})$ for all $v, v' \in R$.*
- *(iii) If $S \in \mathrm{Hom}(v, w)$ and $T \in \mathrm{Hom}(w, x)$ then $ST \in \mathrm{Hom}(v, x)$.*
- *(iv) For any $T \in \mathrm{Hom}(v, v')$ we have $T^* \in \mathrm{Hom}(v', v)$.*
- *(v) If $H_v = H_{v'}$ and $\mathrm{id}_v \in \mathrm{Hom}(v, v')$ then $v = v'$.*
- *(vi) If $S \in \mathrm{Hom}(v, v')$ and $T \in \mathrm{Hom}(w, w')$ then we have $S \otimes T \in \mathrm{Hom}(v \otimes w, v' \otimes w')$.*
- *(vii) We have associativity of $\otimes$ on $R$, i.e. for any $v, v', v'' \in R$ it holds that $(v \otimes v') \otimes v'' = v \otimes (v' \otimes v'')$.*
- *(viii) The trivial representation $1$ is in $R$ with $H_1 = \mathbb{C}$ and $1 \otimes v = v \otimes 1 = v$.*

*It is also complete, meaning that the following conditions also hold:*

- *(ix) For any $v \in R$ and any unitary $w : H_v \to K$ for a Hilbert space $K$, there exists $v' \in \mathrm{Rep}\, G$ such that $H_{v'} = K$ and $w \in \mathrm{Hom}(v, v')$.*
- *(x) For any $v \in R$ and any orthogonal projection $p \in \mathrm{Hom}(v, v)$ there exists $v' \in R$ such that $H_{v'} = pH_v$ and $i \in \mathrm{Hom}(v', v)$ where $i$ is the embedding $H_{v'} \to H_v$.*
- *(xi) For any $v, v' \in R$ there exists a $w \in R$ such that $H_w = H_v \oplus H_{v'}$ and the canonical embeddings $H_v \to H_v \oplus H_{v'}$ and $H_{v'} \to H_v \oplus H_{v'}$ are in $\mathrm{Hom}(v, w)$ and $\mathrm{Hom}(v', w)$ respectively.*

*In addition to the above properties, $R$ is rigid in the sense that for each $v \in R$ there is a conjugate $\overline{v} \in R$ satisfying that there exist intertwiners $S \in \mathrm{Hom}(1, v \otimes \overline{v})$ and $T \in \mathrm{Hom}(1, \overline{v} \otimes v)$ such that*

$$(S^* \otimes \mathrm{id}_v) \cdot (\mathrm{id}_v \otimes T) = \mathrm{id}_v\,, \text{ and } (T^* \otimes \mathrm{id}_{\overline{v}}) \cdot (\mathrm{id}_{\overline{v}} \otimes S) = \mathrm{id}_{\overline{v}}\,.$$

*Moreover, we have that the set consisting of the fundamental corepresentation and its conjugate $\{u, \overline{u}\}$ generates $R$. Here, we say a finite subset $Q \subseteq R$ generates $R$ if for any $s \in R$, there are morphisms $b_k \in \mathrm{Hom}(q_1^{(k)} \cdots q_{n_k}^{(k)}, s)$, $k = 1, \ldots, m$ for some $q_1^{(k)}, \ldots, q_{n_k}^{(k)} \in Q$ such that $\sum_k b_k b_k^* = \mathrm{id}_s \in \mathrm{Hom}(s, s)$.*

We have now seen what the representation category of a compact matrix quantum group looks like. In the spirit of Tannaka-Krein duality, we will now see that any category of this form will in turn yield a compact matrix quantum group.

1.3.20. THEOREM (Tannaka-Krein duality for CMQGs, [90]). *Let a concrete monoidal $W^*$-category be given by $\mathcal{R} = (R, \otimes, \{H_v\}_{v\in R}, \{\mathrm{Hom}(v, v')\}_{v,v'\in R})$ with an object $u \in R$ such that $\{u, \overline{u}\}$ generates $R$. Then there exists a CMQG $G = (C(G), u)$ such that $\overline{\mathcal{R}} = \mathrm{Rep}(G)$, where $\overline{\mathcal{R}}$ is the completion of $\mathcal{R}$.*

1.3.21. REMARK. A class of categories in the sense of the above theorem can be constructed using partitions. Using this together with Tannaka-Krein duality, easy quantum groups were defined in [10] as those compact matrix quantum groups lying between $S_n$ and $O_n^+$ whose representation category arises as a partition category. These have been completely classified in [71]. This approach has been developed further and studied in [57], [58], [59], [60] by adding colours to the partitions and in [20], [35] by adding a third dimension to the partitions.

## 1.4. Nonlocal Games

In this section we introduce the concept of nonlocal games to study the advantages that using quantum resources can give over just using classical resources.

1.4.1. DEFINITION. A two-party *nonlocal game* includes a verifier and two players, who are by convention called Alice and Bob, who devise a cooperative strategy. The game is defined by finite input sets $X_A, X_B$ and finite output sets $Y_A, Y_B$, which are associated to Alice and Bob respectively, a Boolean predicate $V\colon X_A \times X_B \times Y_A \times Y_B \to \{0, 1\}$ and a distribution $\pi$ on $X_A \times X_B$.

In the game, the verifier samples an input $(x_A, x_B) \in X_A \times X_B$ using the distribution $\pi$ and sends $x_A$ to Alice and $x_B$ to Bob. The players then respond with $y_A$ and $y_B$ respectively. The game is said to be won if $V(x_A, x_B, y_A, y_B) = 1$.

In a nonlocal game, the players can devise a strategy beforehand, but cannot communicate after receiving the input. We distinguish the strategies based on whether the players are given access to quantum resources or not. In order to describe the probabilistic nature of strategies, one usually describes them using correlations of the potential outputs given the inputs.

1.4.2. DEFINITION. Let a nonlocal game be given by $\mathcal{G} = (X_A, X_B, Y_A, Y_B, V, \pi)$.

- A *deterministic strategy* is determined by two functions

$$f_A : X_A \to Y_A$$
$$f_B : X_B \to Y_B,$$

one for Alice and one for Bob completely specifying the answers for given inputs. The corresponding correlations are of the form

$$p(y_A, y_B \mid x_A, x_B) = \begin{cases} 1 \text{ if } y_A = f_A(x_A),\, y_B = f_B(x_B), \\ 0 \text{ otherwise.} \end{cases}$$

- A *classical strategy* is determined by correlations of the form

$$p = \sum_i \lambda_i p_i$$

where $\sum_i \lambda_i = 1$ with $\lambda_i > 0$ and all $p_i$ are correlations coming from deterministic classical strategies. In particular, every deterministic strategy is a classical strategy. Those classical strategies that are not deterministic are referred to as *non-deterministic classical strategies*.

In order to describe quantum strategies, we need to introduce some more notions.

1.4.3. Definition. The state space of a quantum system is given by a Hilbert space $H$. A *(pure) state* in $H$ is a unit vector $\psi \in H$. If a system is in state $\psi$, one can extract information from it by performing measurements on $\psi$. This can be modeled using Positive Operator-Valued Measures (POVMs). A *POVM* $\mathcal{M}$ is a family of self-adjoint positive linear operators on $H$

$$\mathcal{M} = \{M_i \in B(H) \mid i \in \{1, \dots, m\}\}$$

for $m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} M_i = 1.$$

The possible outcomes of the measurements are labelled by the indices of the operators and the probability of obtaining outcome $i$ is given by

$$P_{\mathcal{M}}(i) = \psi^* M_i \psi.$$

The conditions that the operators sum to one and that they are all positive ensure that this defines a probability distribution. If all the operators $M_i$ are orthogonal projections then we call $\mathcal{M}$ *projective* or a *projective valued measure* (PVM).

1.4.4. Definition. Let $H_1$ and $H_2$ be two Hilbert spaces. If a quantum system has state space $H_1 \otimes H_2$, we say that it is a *joint system*. A state $\psi \in H_1 \otimes H_2$ is said to be *separable* if it can be written as $\psi = \psi_1 \otimes \psi_2$ for $\psi_1 \in H_1$ and $\psi_2 \in H_2$ and *entangled* otherwise. An example of an entangled state is the so-called *maximally entangled state*

$$\psi_d := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d,$$

where $e_i$ is the $i$-th standard basis vector.

1.4.5. DEFINITION. Let a nonlocal game be given by $\mathcal{G} = (X_A, X_B, Y_A, Y_B, V, \pi)$. For a *quantum commuting strategy* Alice and Bob share a Hilbert space $H$ and a pure state $\psi \in H$. Moreover, they both have families of POVMs

$$\mathcal{A}_{x_A} = \{A_{x_A,y_A} \mid y_A \in Y_A\} \text{ for each } x_A \in X_A$$
$$\mathcal{B}_{x_B} = \{B_{x_B,y_B} \mid y_B \in Y_B\} \text{ for each } x_B \in X_B$$

satisfying that all of Alice's measurement operators commute with all of Bob's measurement operators. Upon receiving a question $x_A$, Alice will then perform the measurement $\mathcal{A}_{x_A}$ and receive an outcome $y_A$, which will be her answer. Bob will act correspondingly with his measurements. The probability of Alice and Bob answering with $y_A$ and $y_B$ upon receiving the questions $x_A$ and $x_B$ is then given by

$$p(y_A, y_B|x_A, x_B) = \psi^* A_{x_A,y_A} B_{x_B,y_B} \psi.$$

1.4.6. REMARK. Sometimes, one only considers the special case where the Hilbert space $H$ is the tensor product of two finite-dimensional Hilbert spaces $H = H_A \otimes H_B$. All of Alice's operators are then of the form $A_{x_A,y_A} \otimes 1$ for $A_{x_A,y_A} \in B(H_A)$ and Bob's operators are of the form $1 \otimes B_{x_B,y_B}$ for $B_{x_B,y_B} \in B(H_B)$. The operators of Alice and Bob then automatically commute. The kind of strategy as detailled above is called *quantum tensor strategy*. One can show that if a quantum tensor strategy uses a separable state then one can not gain any advantage over classical strategies. In this thesis, we only consider the more general setting of quantum commuting strategies.

When studying nonlocal games there are many things one can consider, such as what is the best possible winning probability a certain family of strategies can achieve? Are there advantages one can gain when playing multiple rounds? We however want to mostly consider so-called perfect strategies and only consider the case where only one round is played.

1.4.7. DEFINITION. Let $\mathcal{G}$ be a nonlocal game. A strategy for $\mathcal{G}$ and its defining correlation $p$ are called *perfect* if it wins for every possible input. In other words, they are perfect if for every losing combination of inputs and outputs, the probability of obtaining this combination according to $p$ is zero:

$$V(x_A, x_B, y_A, y_B) = 0 \implies p(y_A, y_B \mid x_A, x_B) = 0.$$

Note that when checking whether there exists a perfect classical strategy, it suffices to only consider deterministic strategies.

1.4.8. LEMMA. *A nonlocal game has a perfect classical strategy if and only if there exists a perfect deterministic strategy for the game*

PROOF. First note that any deterministic strategy is in particular a classical strategy and therefore one implication is immediate. For the other implication, let a perfect classical strategy be given by the correlation $p = \sum_i \lambda_i p_i$. We claim that each of the correlations $p_i$ must belong to a perfect deterministic strategy. Indeed, assuming that one of the $p_i$ did not come from a perfect strategy would imply that there exist some inputs and outputs such that

$$p_i(y_A, y_B \mid x_A, x_B) > 0 \text{ for which } V(x_A, x_B, y_A, y_B) = 0.$$

But since all $\lambda_i$ must be larger than zero we then get

$$p(y_A, y_B \mid x_A, x_B) \geq \lambda_i p_i(y_A, y_B \mid x_A, x_B) > 0,$$

which is a contradiction to the assumption that $p$ belonged to a perfect strategy. Therefore, each $p_i$ must be a perfect correlation and in particular at least one perfect deterministic strategy exists. □

1.4.9. Example. An important example of nonlocal games is the *CHSH*-game, introduced in [25] as an experimental verification of the existence of entanglement. It goes as follows: The verifier uniformly and independently samples two bits $x, y \in \{0, 1\}$ and gives $x$ to Alice and $y$ to Bob. Alice and Bob now respond again with two bits $a, b \in \{0, 1\}$ and win, if the following equation holds:

$$x \cdot y = a \oplus b,$$

where $\oplus$ denotes addition modulo 2. One can easily see that the optimal winning probability for this game when only using classical strategies is 75%.

Using the following quantum strategy however, one can get a higher winning probability: Alice and Bob share the state $\psi = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and associate the following measurements to the possible questions:

$$\begin{aligned}
\mathcal{A}_0 &= \{A_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_{01} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}, \\
\mathcal{A}_1 &= \{A_{10} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, A_{11} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\}, \\
\mathcal{B}_0 &= \{B_{00} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} \end{pmatrix}, B_{01} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix}\}, \\
\mathcal{B}_1 &= \{B_{10} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} \end{pmatrix}, B_{11} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix}\}.
\end{aligned}$$

Here we labelled the elements of the POVMs with their corresponding question and answers, i.e. upon being asked the question "0", Alice will perform the measurement $\mathcal{A}_0$ and receive either 0 or 1 as answer, with probability $\psi^* A_{00} \otimes 1\psi$ or $\psi^* A_{01} \otimes 1\psi$ respectively. Similar things hold for Bob. Assuming now, that both Alice and Bob receive the question $x = 0 = y$, we can compute the result of the measurement

$\mathcal{A}_0 \otimes \mathcal{B}_0$ and get

$$\begin{aligned}
p(0,0\,|\,0,0) &= \psi^* A_{00} \otimes B_{00}\psi \\
&= \frac{1}{2}(e_1 \otimes e_1 + e_2 \otimes e_2)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} \end{pmatrix} (e_1 \otimes e_1 + e_2 \otimes e_2) \\
&= \frac{1}{2}(e_1 \otimes e_1)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} \end{pmatrix} e_1 \otimes e_1 + 0 \\
&= \frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{2\sqrt{2}}).
\end{aligned}$$

The probability of both Alice and Bob answering with "0" is therefore given by $p(0,0\,|0,0) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2\sqrt{2}})$. Doing the same computation for $\psi^* A_{01} \otimes B_{01}\psi$ yields the same probability for both Alice and Bob answering with "1" and we get all in all that the probability of Alice and Bob winning, i.e. answering with the same answer, is $\frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$. Going through all other possible combinations of questions one can now compute that the probability to win always stays at $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ and we see that the winning probability when using this quantum strategy is thus higher than the best possible winning probability with any classical strategy.

1.4.10. Example. In [2], a nonlocal game was introduced, that captures the notion of graph isomorphism when using classical strategies and therefore allows a natural extension of the notion of graph isomorphism, by considering quantum strategies. We present here a variant of the game, where the input and output graphs are fixed, rather than taking as question and answer sets both graphs simultaneously. However, the analysis of the game remains largely the same. Given two graphs, $\Gamma_1$ and $\Gamma_2$ both on $n \in \mathbb{N}$ vertices, the $(\Gamma_1, \Gamma_2)$-*isomorphism game* is defined as follows: The verifier uniformly samples two vertices $x_A, x_B \in V(\Gamma_1)$ and gives them to Alice and Bob respectively. Alice and Bob then respond with vertices $y_A, y_B \in V(\Gamma_2)$. Alice and Bob win the game, if two conditions are met:

(i) $x_A \sim x_B \iff y_A \sim y_B$

(ii) $x_A = x_B \iff y_A = y_B$.

Also in [2] it was shown that there exist graphs that are quantum isomorphic that are however not classically isomorphic.

The isomorphism game however is not the only nonlocal game on graphs. In [56], the authors define a graph homomorphism game $\mathrm{Hom}(\Gamma_1, \Gamma_2)$ as a generalisation of the graph coloring game for which a perfect classical strategy exists if and only if there exists a homomorphism from $\Gamma_1$ to $\Gamma_2$. They use this game to define quantum homomorphisms of graphs and to then define new quantum variants of different established graph parameters, such as the quantum chromatic number.

In this thesis, we are mostly concerned with a certain type of game, namely synchronous games, where the rules of the game are symmetric for the two players and where in particular, when asked the same question they must answer with the same answer.

1.4.11. Definition. A nonlocal game is called *synchronous*, if the input sets and the output sets are the same for both players, i. e. if $X = X_A = X_B$, $Y = Y_A = Y_B$ and if additionally both players must give the same answer upon receiving the same question, i.e. $V(y_1, y_2, x, x) = 0$ for all $x \in X$ and $y_1 \neq y_2 \in Y$.

We formulate these games as synchronous games since for synchronous games, we have additional information about their perfect quantum commuting strategies. The following result is part of the analysis in [69] and was formulated as a lemma in [2].

1.4.12. Lemma. *Let a synchronous game with input sets $X$, output sets $Y$ and verifier function $V$ be given. Then a perfect quantum commuting strategy for the game exists if and only if there exist a unital $C^*$-algebra $\mathcal{A}$ admitting a faithful tracial state and projections $E_{xy} \in \mathcal{A}$ for $(x, y) \in X \times Y$ satisfying:*

*(i)* $\sum_{y \in Y} E_{xy} = 1$.
*(ii)* $E_{xy} E_{x'y'} = 0$ *if* $V(y, y' \mid x, x') = 0$.

1.4.13. Remark. Recently, a generalisation of nonlocal games has been studied [18], [79] no longer only allowing classical input and output sets but now also allowing to replace either or both of those to be a “quantum set”. When given finite sets $X$ and $Y$ we denote by $M_{XY}$ the matrices over $\mathbb{C}$ indexed by the elements of $X \otimes Y$. One can then quantise the input and output sets of the nonlocal game by looking at the projection lattices in $M_{XY}$:

$$\mathcal{P}_{XY} \coloneqq \{P \in M_{XY} \mid P \text{ is a projection}\}.$$

Looking only at the diagonal matrices in $M_{XY}$, denoted by $D_{XY}$, one obtains the diagonal projection lattices which correspond to the classical sets:

$$\mathcal{P}_{XY}^{cl} \coloneqq \{P \in D_{XY} \mid P \text{ is a projection}\}.$$

For projections $P$ and $Q$, one moreover defines the *join* $P \vee Q$ of $P$ and $Q$ as the projection with the range being the span of the union of the ranges of $P$ and $Q$. Quantum nonlocal games are now identified with their rule function and for finite sets $X, Y, A$ and $B$ we say a map $\varphi : \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ is a quantum nonlocal game if $\varphi(0) = 0$ and if it is *join continuous*, i.e. $\varphi(\bigvee_{i \in I} P_i) = \bigvee_{i \in I} \varphi(P_i)$ for an index set $I$. A classical nonlocal game is then a join continuous map $\varphi : \mathcal{P}_{XY}^{cl} \to \mathcal{P}_{AB}^{cl}$ with $\varphi(0) = 0$ and we can also define classical to quantum (resp. quantum to classical) nonlocal games as join continuous maps $\varphi : \mathcal{P}_{XY}^{cl} \to \mathcal{P}_{AB}$ (resp. $\varphi : \mathcal{P}_{XY} \to \mathcal{P}_{AB}^{cl}$) satisfying $\varphi(0) = 0$. In [79] the authors introduce these games and study quantum non-signalling strategies for these quantum nonlocal games and in [18] the authors study possibilities to generalise the concept of synchronous games to the field of quantum nonlocal games.

CHAPTER 2

# Quantum Automorphism Groups of Graphs

In this section we will review some of the existing results around quantum automorphism groups of graphs and also collect some techniques that can be helpful when studying the quantum automorphism group of a concrete graph.

## 2.1. Basic Definitions

Quantum automorphism groups of graphs were defined by Banica in [4].

2.1.1. Definition. Given a graph $\Gamma$ on $n$ vertices, its quantum automorphism group $G^+_{aut}(\Gamma)$ is the compact matrix quantum group $\big(C(G^+_{aut}(\Gamma)), u\big)$, where $C(G^+_{aut}(\Gamma))$ is the universal $C^*$-algebra with generators $u_{ij}$, $1 \leq i, j \leq j$ subject to the following relations:

$$u_{ij} = u_{ij}^* = u_{ij}^2 \qquad 1 \leq i, j \leq n \tag{2.1.1}$$

$$\sum_{k=1}^n u_{ik} = 1 = \sum_{k=1}^n u_{ki} \qquad 1 \leq i \leq n \tag{2.1.2}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0 \qquad i \sim k, j \not\sim l \tag{2.1.3}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0 \qquad i \not\sim k, j \sim l. \tag{2.1.4}$$

To see the connection to the classical automorphism group of $\Gamma$, the relations above can equivalently be written as

$$C(G^+_{aut}(\Gamma)) = C^*(u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = 1 = \sum_{k=1}^n u_{ki}, 1 \leq i \leq n, A_\Gamma u = uA_\Gamma),$$

where $A_\Gamma$ is the adjacency matrix of $\Gamma$. The equivalence of the two definitions has e.g. been shown in Proposition 2.1.3 in [75].

2.1.2. Remark. In [13], Bichon gave a slightly different definition of quantum automorphism groups of graphs: in his definition, the relations from 2.1.1 stay the same but there is the additional relation

$$u_{ij}u_{kl} = u_{kl}u_{ij} \text{ for all } i \sim k, j \sim l.$$

This defines a quantum subgroup of Banica's quantum automorphism group. In this thesis, we will only be considering Banica's version.

As can be seen immediately from the definition, any quantum automorphism group of a graph on $n$ vertices is a quantum subgroup of the quantum symmetric group $S_n^+$. Moreover, taking the above definition for a given graph $\Gamma$ and adding commutation relations, one can see that

$$C(G_{aut}^+(\Gamma))/\langle u_{ij}u_{kl} = u_{kl}u_{ij}\,\forall i,j,k,l\rangle = C(G_{aut}(\Gamma)),$$

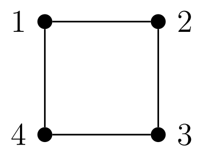
and therefore we have $G_{aut}(\Gamma) \subseteq G_{aut}^+(\Gamma)$. The interesting question to study is now when the relations of Definition 2.1.1 already imply commutativity and when they do not, i.e. in which cases the quantum automorphism group of the graph is actually bigger than the classical one.

Since for $n \leq 3$ the relations of $S_n^+$ already imply commutativity of all generators, this question is only sensible as soon as $n \geq 4$. This leads to the following definition.

2.1.3. Definition. Let $\Gamma$ be a graph. We say $\Gamma$ *has quantum symmetries* if $C(G_{aut}^+(\Gamma))$ is noncommutative. On the other hand we say $\Gamma$ *has no quantum symmetries*, if $C(G_{aut}^+(\Gamma))$ is commutative, i.e. if $G_{aut}^+(\Gamma) = G_{aut}(\Gamma)$.

2.1.4. Example.

- The cycle on 4 vertices $C_4$ has quantum symmetries. To see this, we will label the vertices of $C_4$ as follows:




  Now taking the matrix

$$\tilde{u} := \begin{pmatrix} p & 0 & 1-p & 0 \\ 0 & q & 0 & 1-q \\ 1-p & 0 & p & 0 \\ 0 & 1-q & 0 & q \end{pmatrix}$$

  in the universal $C^*$-algebra $A := C^*(p, q, 1 \mid p, q \text{ projections})$ one can get a surjective $*-$homomorphism from $C(G_{aut}^+(C_4))$ onto $A$ by mapping the generator matrix $u$ of $C(G_{aut}^+)$ to $\tilde{u}$. Since $pq \neq qp$ in $A$ this proves the noncommutativity. One can show that $G_{aut}^+(C_4) = H_2^+$.
- For any other cycle $C_n$ with $n \neq 4$ one can see that it does not have quantum symmetries, i.e. all $C(G_{aut}^+(C_n))$ are commutative, as was shown by Banica in Corollary 4.1 in [4].
- In [74] it was shown that the Petersen graph, an important example for many graph properties, does not have quantum symmetries.

## 2.2. Tools for Computing Quantum Symmetries

In this section we collect some lemmas that can be helpful when trying to decide for a concrete graph, whether or not it has quantum symmetries. Some of them come from the PhD thesis of Schmidt [75] while others are from an article by the author of the present thesis [73].

The first lemma is essential in a lot of the proofs of commutativity of quantum automorphism groups of graphs in this work. It has appeared for example in [75].

2.2.1. Lemma. *Let $A$ be a $C^*$-algebra and $a, b \in A$ be self-adjoint elements. If we have*

$$ab = aba$$

*then $a$ and $b$ commute.*

Proof. Since $aba$ is self-adjoint the claim follows immediately. □

The next lemma provides an easy to check criterion that implies that a given graph has quantum symmetries.

2.2.2. Lemma (Theorem 3.1.2 in [75]). *Let $\Gamma$ be a graph. If there exist two non-trivial, disjoint automorphisms in $Aut(\Gamma)$, then $\Gamma$ has quantum symmetries and its quantum automorphism group is infinite dimensional.*

The above lemma however is not a complete characterisation of graphs with quantum symmetries, as there are graphs without disjoint automorphisms that do have quantum symmetries, as has been for example shown in [75].

The next lemma is a quantisation of the fact that any automorphism of a graph has to preserve the distance of any pair of vertices.

2.2.3. Lemma (Lemma 3.2.2 in [75]). *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If we have $d(i,k) \neq d(j,l)$, then $u_{ij}u_{kl} = 0$.*

The following two lemmas are quite technical, however they allow the simplification of some computations, and in particular their conditions can easily be checked by a computer, which can allow a certain amount of automation in the computation of the existence of quantum symmetries.

2.2.4. Lemma (Lemma 3.2.8 in [75]). *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $d(i,k) = d(j,l) = m$. Let $q$ be a vertex with $d(j,q) = s$, $d(q,l) = t$ and $u_{kl}u_{aq} = u_{aq}u_{kl}$ for all $a$ with $d(a,k) = t$. Then*

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;\, d(l,p)=m,\\ d(p,q)=s}} u_{ip}.$$

*In particular, if we have $m = 2$ and if $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ holds, then choosing $s = t = 1$ implies*

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;\, d(p,l)=2\\ p\sim q}} u_{ip}.$$

2.2.5. Lemma (Lemma 3.2.9 in [75]). *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $d(i,k) = d(j,l) = m$ and let $p \neq j$ be a vertex with $d(p,l) = m$. Let $q$ be a vertex with $d(q,l) = s$ and $d(j,q) \neq d(q,p)$. Then*

$$u_{ij}\left(\sum_{\substack{t;\, d(t,j)=d(t,p)=m\\ d(t,q)=s}} u_{kt}\right)u_{ip} = 0.$$

*Especially, if $l$ is the only vertex satisfying $d(l,q) = s$, $d(l,j) = m$ and $d(l,p) = m$, we obtain $u_{ij}u_{kl}u_{ip} = 0$.*

The following lemma is a relatively simple statement about a vanishing monomial that first appeared in an article by the author of this thesis [73].

2.2.6. Lemma (Lemma 2.1.6 in [73]). *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})$ be the generators of $G^+_{aut}(\Gamma)$. Let $i,j,k,l,p,q \in V(\Gamma)$ be vertices such that $d(p,q) \neq d(j,q)$ and $u_{kl}u_{rq} = u_{rq}u_{kl}$ for any vertex $r \in V(\Gamma)$ with $d(i,r) = d(p,q)$. Then it holds that*

$$u_{ij}u_{kl}u_{ip} = 0.$$

Proof. We observe that

$$\begin{aligned} u_{ij}u_{kl}u_{ip} &= u_{ij}u_{kl}\sum_{\substack{r\in V(\Gamma)\\ d(i,r)=d(p,q)}} u_{rq}\, u_{ip} \\ &= \underbrace{u_{ij}\sum_{\substack{r\in V(\Gamma)\\ d(i,r)=d(p,q)}} u_{rq}}_{=0}\, u_{kl}u_{ip} \\ &= 0. \end{aligned}$$

Here, the commutation of $u_{kl}$ and $u_{rq}$ is by assumption and the fact that $u_{ij}u_{rq} = 0$ is due to the fact that $d(i,r) = d(p,q) \neq d(j,q)$ by assumption.

□

The next lemma is an adaptation of Lemma 3.2.4 in [75] for a less general case, the proof however is the same.

2.2.7. Lemma (Lemma 3.2.4 in [75]). *Let $\Gamma$ be a graph and $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $j_1, l_1, j_2, l_2$ be vertices in $\Gamma$ such that $d(j_1,l_1) = d(j_2,l_2)$. If we have $u_{aj_1}u_{bl_1} = u_{bl_1}u_{aj_1}$ for all $a,b \in V(\Gamma)$ and there exists an automorphism $\varphi \in G_{aut}(\Gamma)$ such that $\varphi(j_1) = j_2$ and $\varphi(l_1) = l_2$, then we get*

$$u_{aj_2}u_{bl_2} = u_{bl_2}u_{aj_2} \text{ for all } a,b \in V(\Gamma).$$

For vertex-transitive graphs the above lemma leads immediately to the following corollary:

2.2.8. COROLLARY. *Let $\Gamma$ be a vertex-transitive graph and $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If there exists a vertex $j_0 \in V(\Gamma)$ and a distance $m$ such that for all vertices $l \in V(\Gamma)$ with $d(j_0,l) = m$ we have commutation of $u_{ij_0}$ and $u_{kl}$ for any vertices $i,k \in V(\Gamma)$, then it already holds that for all vertices $i,j,k,l \in V(\Gamma)$ we have*

$$d(j,l) = m \implies u_{ij}u_{kl} = u_{kl}u_{ij}.$$

*In particular, if $u_{ij_0}$ commutes with all other generators $u_{kl}$ of $C(G^+_{aut}(\Gamma))$, then $\Gamma$ does not have quantum symmetries.*

PROOF. Let $j', l' \in V(\Gamma)$ be vertices of $\Gamma$ such that $d(j',l') = m$. Since $\Gamma$ is vertex-transitive, there exists an automorphism $\varphi$ of $\Gamma$ that maps $j_0$ to $j'$. Since $\varphi$ is an automorphism, the preimage of $l_0 := \varphi^{-1}(l')$ of $l'$ must satisfy

$$d(j_0,l_0) = d(j',l') = m.$$

By assumption, we know that $u_{ij_0}u_{kl_0} = u_{kl_0}u_{ij_0}$ for any $i,k \in V(\Gamma)$. Applying Lemma 2.2.7, we get that also $u_{\varphi(i)j'}$ and $u_{\varphi(k)l'}$ commute. Since $i$ and $k$ were arbitrary, we also get $u_{ij'}u_{kl'} = u_{kl'}u_{ij'}$ for arbitrary $i,k \in V(\Gamma)$. □

As seen in Lemma 2.2.3, the distance between vertices matters when looking at the generators of the quantum automorphism group of a graph. The simplest case to consider when looking at distances is the neighbour. Because of this, a few lemmas were found concerning common neighbours of vertices.

The following two lemmas provide an easy way to see the commutation of all generators for adjacent vertices by looking at the structure of the graph, namely the existence of quadrangles and triangles.

2.2.9. LEMMA (Lemma 3.2.5 in [75]). *Let $\Gamma$ be an undirected graph that does not contain any quadrangle. Then for vertices $i \sim k$ and $j \sim l$ the generators $u_{ij}$ and $u_{kl}$ commute.*

2.2.10. LEMMA (Lemma 3.2.6 in [75]). *Let $\Gamma$ be an undirected graph such that adjacent vertices have exactly one common neighbour. Then for vertices $i \sim k$ and $j \sim l$ the generators $u_{ij}$ and $u_{kl}$ commute.*

The above lemma can be generalised a bit.

2.2.11. LEMMA (Lemma 2.3.3 in [73]). *Let $\Gamma = (V,E)$ be an undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $i,j,k,l \in V$ be vertices of $\Gamma$ such that $(i,k) \in E$ and $(j,l) \in E$. Let moreover $i$ and $k$ have exactly one common neighbour $p$ and let it hold that for any two vertices from the set $\{i,k,p\}$ the third vertex from the set is the only common neighbour of the two, i.e. the only common neighbour of $i$ and $p$ is $k$ and the only common neighbour of $k$ and $p$ is $i$. Let $j$ and $l$ also have exactly one common neighbour $q$ and let the same as above hold for the set of vertices $\{j,l,q\}$.*

*Then* $u_{ij}u_{kl} = u_{kl}u_{ij}$.

PROOF. The proof is the same as the one of Lemma 2.2.10 which was given in the PhD thesis of Schmidt [75]:

Using the relations of $C(G^+_{aut}(\Gamma))$, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{a;(l,a)\in E} u_{ia}.$$

We want to show that $u_{ij}u_{kl}u_{ia} = 0$ for $a \neq j$, $(a,l) \in E$. First, we consider $q$, the unique common neighbour of $j$ and $l$. We note, that for any vertex $b \neq l$ we have $(j,b) \notin E$ or $(q,b) \notin E$, since $l$ is also the only common neighbour of $j$ and $q$. We deduce

$$u_{ij}u_{kl}u_{iq} = u_{ij}\left(\sum_b u_{kb}\right)u_{iq} = u_{ij}1u_{iq} = u_{ij}u_{iq} = 0.$$

Let now $a \notin j, q$ and $(a,l) \in E$. Then it holds that

$$u_{ij}u_{kl} = u_{ij}\left(\sum_b u_{pb}\right)u_{kl} = u_{ij}u_{pq}u_{kl}$$

since $p$ is the only common neighbour of $i$ and $k$ while $q$ is the only common neighbour of $j$ and $l$. As $j$ is the only common neighbour of $l$ and $q$ and we have that $(a,l) \in E$, we deduce that $(a,q) \notin E$. We thus get

$$0 = u_{pq}u_{ia} = u_{pq}\left(\sum_c u_{cl}\right)u_{ia} = u_{pq}u_{kl}u_{ia}$$

since $(i,p) \in E$ but $(a,q) \notin E$ and since $k$ is the only common neighbour of $p$ and $i$.

Using the two equations we deduced above, we get

$$u_{ij}u_{kl}u_{ia} = u_{ij}u_{pq}u_{kl}u_{ia} = 0.$$

We thus get for all $a \neq j$ that

$$u_{ij}u_{kl}u_{ia} = 0.$$

Finally, we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(\sum_a u_{ia}\right) = u_{ij}u_{kl}u_{ij}$$

and thus we get by Lemma 2.2.1 that $u_{ij}u_{kl} = u_{kl}u_{ij}$. $\square$

We introduce a new notation to make some statements and computations more compact:

2.2.12. DEFINITION. Given a graph $\Gamma$ and vertices $i, j$ of $\Gamma$, we denote by $\mathrm{CN}(i,j)$ the set of all common neighbours of $i$ and $j$, i.e. all vertices $k$ of $\Gamma$ such that both $i \sim k$ and $j \sim k$ hold.

The following lemma again yields an easy to check criterion for a certain product of generators to be zero.

2.2.13. LEMMA (Lemma 2.3.5 in [73]). *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Let $i,j,k,l \in V$ be vertices in $\Gamma$.*
*If $|\mathrm{CN}(i,k)| \neq |\mathrm{CN}(j,l)|$ then we already have*

$$u_{ij}u_{kl} = 0.$$

PROOF. We set $a = |\mathrm{CN}(i,k)|$ and $b = |\mathrm{CN}(j,l)|$ and observe:

$$\begin{aligned} a \cdot u_{ij}u_{kl} &= \sum_{p\in \mathrm{CN}(i,k)} \underbrace{u_{ij} \sum_{q\in \mathrm{CN}(j,l)} u_{pq}u_{kl}}_{=u_{ij}u_{kl}} \\ &= u_{ij} \sum_{p\in \mathrm{CN}(i,k)} \sum_{q\in \mathrm{CN}(j,l)} u_{pq}u_{kl} \\ &= \sum_{q\in \mathrm{CN}(j,l)} \underbrace{u_{ij} \sum_{p\in \mathrm{CN}(i,k)} u_{pq}u_{kl}}_{=u_{ij}u_{kl}} \\ &= b \cdot u_{ij}u_{kl}. \end{aligned}$$

Since by assumption we had $a \neq b$, we get that $u_{ij}u_{kl} = 0$. ☐

This lemma leads to the following corollaries.

2.2.14. COROLLARY. *If $i,j,k,l,p$ are vertices in $\Gamma$ and $|\mathrm{CN}(j,l)| \neq |\mathrm{CN}(l,p)|$ then*

$$u_{ij}u_{kl}u_{ip} = 0.$$

PROOF. If $|\mathrm{CN}(i,k)| \neq |\mathrm{CN}(j,l)|$ then we have $u_{ij}u_{kl} = 0$ and thus also $u_{ij}u_{kl}u_{ip} = 0$. Otherwise we have $|\mathrm{CN}(i,k)| = |\mathrm{CN}(j,l)| \neq |\mathrm{CN}(l,p)|$ and thus $u_{kl}u_{ip} = 0$ and thus $u_{ij}u_{kl}u_{ip} = 0$. ☐

2.2.15. COROLLARY. *Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. If $i,j,k,l \in V$ are vertices of $\Gamma$ with $(i,k) \in E$ and $(j,l) \in E$ such that $i$ and $k$ have at least one common neighbour, but $j$ and $l$ do not have a common neighbour, then*

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0.$$

*In other words, if $i$ and $k$ are part of the same triangle but $j$ and $l$ are not, then the product of $u_{ij}$ and $u_{kl}$ is zero.*

PROOF. If $i$ and $k$ are part of the same triangle, then in particular $|\mathrm{CN}(i,k)| > 0$, while $|\mathrm{CN}(j,l)| = 0$, since they are not in the same triangle. Applying Lemma 2.2.13, we get $u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$. ☐

**2.2.1. A strategy for deciding quantum symmetries.** In this section we want to give a short checklist of steps to take when one has a given graph $\Gamma$ and wants to decide, whether it has quantum symmetries.

1. First, one should check, whether $\Gamma$ has disjoint automorphisms. If yes, then by Lemma 2.2.2 it has quantum symmetries.
2. If $\Gamma$ does not have disjoint automorphisms, it is not clear, whether or not $\Gamma$ has quantum symmetries. However, it seems to be a good idea to first try to show that it does not have quantum symmetries, i.e. that $C(G^+_{aut}(\Gamma))$ is commutative. The following can serve as a strategy for this.
   2.1. By Lemma 2.2.3, it suffices to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ only for vertices that fulfill $d(i,k) = d(j,l)$. One can thus fix a distance $m = d(i,k) = d(j,l)$ and try to show commutativity for vertices in this distance.
   2.2. For $m = 1$, the first thing to check is whether Lemma 2.2.9 or Lemma 2.2.10 are applicable, because these already yield commutativity for this case.
   2.3. In the next step, one can check whether using Lemmas 2.2.4 or 2.2.5 one can get $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ by showing that $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$. This is a task that can easily be delegated to a computer and can therefore be applied to all possible combinations of generators easily.
   2.4. If the results of Lemmas 2.2.4 and 2.2.5 only yield partial results, i.e. they only show $u_{ij}u_{kl}u_{ip} = 0$ for some of the vertices $p$, one can try to use Corollary 2.2.14 to show it for the rest.
   2.5. If at any point one has shown $u_{ij}u_{kl} = u_{kl}u_{ij}$ for any generators, one can use Lemma 2.2.7 to propagate this to other pairs of generators.
   2.6. Any remaining pairs of vertices will have to be considered individually, potentially using a combination of Lemmas 2.2.4, 2.2.5, 2.2.7 and Corollary 2.2.14.

## 2.3. Some Known Results

We collect some previous results on quantum automorphism groups of graphs. The choice of results presented is however somewhat arbitrary and not exhaustive.

Shortly after defining quantum automorphism groups of graphs, Banica and Bichon computed the quantum automorphism group for all vertex-transitive graphs on $\leq 11$ vertices except the Petersen graph in [6]. Later, in [74] Schmidt showed that the Petersen graph does not have quantum symmetries. Meanwhile Chassaniol computed the quantum automorphism groups of all vertex-transitive graphs on 13 vertices in [22] and [23].

2.3.1. Lemma ([6], [22], [23], [74]). *The quantum automorphism groups, and therefore also the existence of quantum symmetries, are known for all vertex-transitive graphs on $\leq 11$ and on 13 vertices.*

In Section 4, we give a more detailed overview over these results on vertex-transitive graphs.

In [8], Banica, Bichon and Collins moreover compute the quantum automorphism groups of cube graphs and in [7] Banica, Bichon and Chenevier show that certain circulant graphs on a prime number of vertices do not have quantum symmetries.

In a recent paper by van Dobben de Bruyn, Nigam Kar, Roberson, Schmidt and Zeman, a different important class of graphs, namely trees, has been completely characterised in terms of their quantum automorphism groups.

2.3.2. THEOREM ([30]). *The class $\mathcal{T}$ of quantum automorphism groups of trees can be constructed inductively in the following way:*

*(i)* $\mathbf{1} \in \mathcal{T}$.
*(ii) If* $\mathbb{G}, \mathbb{H} \in \mathcal{T}$, *then also* $\mathbb{G} * \mathbb{H} \in \mathcal{T}$.
*(iii) If* $\mathbb{G} \in \mathcal{T}$, *then also* $\mathbb{G} \wr_* S_n^+ \in \mathcal{T}$.

The proof they give is constructive and they also give a polynomial-time algorithm that can calculate the quantum automorphism group of any tree.

Another important recent result is a paper by van Dobben de Bruyn, Roberson and Schmidt that answers a question that has long been of interest: are there asymmetric graphs that have quantum symmetries?

2.3.3. THEOREM ([31]). *There is an infinite sequence of non-isomorphic graphs* $\Gamma_i$ *for* $i \in \mathbb{N}$ *such that* $\Gamma_i$ *has quantum symmetry but* $G_{aut}(\Gamma_i)$ *is the trivial group for all* $i \in \mathbb{N}$.

The question whether there are asymmetric graphs with quantum symmetry can be extended to the question whether there are any groups such that when a graph has this group as automorphism group, the graph can not have quantum symmetries. Also this question was answered in the same paper.

2.3.4. THEOREM ([31]). *Let $G$ be a finite group. Then there exist graphs* $\Gamma_1$ *and* $\Gamma_2$ *such that* $G_{aut}(\Gamma_i) \cong G$ *for* $i = 1, 2$ *and* $G^+_{aut}(\Gamma_1) = G_{aut}(\Gamma_1)$ *but* $\Gamma_2$ *has quantum symmetries.*

In [51], Lupini, Mančinska and Roberson show a quantum analogue of the statement by Erdős and Rényi in [34] that almost all graphs have trivial automorphism group.

2.3.5. THEOREM ([51]). *Let* $\Gamma$ *be a random graph on $n$ vertices. The probability that* $\Gamma$ *has non-trivial quantum automorphism group goes to zero as $n$ goes to infinity.*

Moreover, Mančinska and Roberson study the intertwiner spaces of quantum automorphism groups of graphs in [55], which have also been studied by Chassaniol in [22], [23].

In [42], Gromada studies quantum automorphisms and quantum isomorphisms of Hadamard matrices and the corresponding Hadamard graphs.

2.3.6. Theorem ([42]). *Let $H$ be a Hadamard matrix of size $N$. If $N \geq 4$, then both $H$ and its corresponding Hadamard graph have quantum symmetries.*

The following result has been independently shown by Gromada [42] and by Chan and Martin [21].

2.3.7. Theorem ([21], [42]).

- *All Hadamard matrices of a fixed size are mutually quantum isomorphic.*
- *All Hadamard graphs of a fixed size are mutually quantum isomorphic.*

In [48], the author of this thesis together with several co-authors used a computer algebra approach to compute for all graphs on 6 or less vertices and for some graphs on 7 vertices, whether or not they have quantum symmetries.

CHAPTER 3

# Computing Noncommutative Gröbner Bases Using OSCAR

This thesis has been supported by the SFB-TRR 195, which has the topic of symbolic tools in mathematics and their application. One of the important results of this research programme is the development of the free and open-source `OSCAR` computer algebra system [28], [66], which is written in the `julia` programming language [12]. `OSCAR` unites several pre-existing computer algebra systems and makes it easy to have the different subsystems communicate with each other. However, there also exist several algorithms that are completely written within `OSCAR`. In this thesis, we employ at certain points a noncommutative Gröbner basis computation that has been implemented in `OSCAR` by Schultz and the author of this thesis. In this chapter, we give a basic introduction to the most important concepts of noncommutative Gröbner basis computation.

## 3.1. The Noncommutative Buchberger Algorithm

In order to computationally handle ideals in polynomial rings, Gröbner bases are the standard way to go. Originally introduced for commutative polynomial rings over fields by Buchberger in [19], their use case has since been extended to also include noncommutative Rings by Mora in [63]. The idea is as follows: one is given a set of generators of the ideal. Using an algorithm of choice, the Buchberger procedure being the first and perhaps most famous one, one transforms these generators into a Gröbner basis, allowing to efficiently decide for any given polynomial, whether it belongs to the ideal.

Implementations of the original Buchberger algorithm for commutative rings are common in a lot of computer algebra systems nowadays. The noncommutative case, while it exists in some computer algebra systems, is not so common. There is for example an implementation of a noncommutative Buchberger procedure in `OSCAR` [28], [66], which is based on the algorithm as it is given in [93] and which the author used for certain computer based computations in this thesis. In this section we will give some very basic definitions and a sketch of the Buchberger procedure for noncommutative polynomial rings and some simple optimizations one can do to improve the running time of the algorithm. These definitions and statements also come from [93].

Throughout this chapter, let $K$ be a field, $X = \{x_1, \dots, x_n\}$ a finite alphabet and let $K\langle X\rangle$ be the free monoid ring generated by $X$ over $K$. By $\langle X\rangle$ we mean the free monoid generated by $X$. We will moreover only be considering two-sided ideals in $K\langle X\rangle$.

If we have a relation $\sigma \subseteq S \times S$ on any set $S$ we will write $a \geq_\sigma b$ or $b \leq_\sigma a$ to mean $(a, b) \in \sigma$. Moreover, if $a \geq_\sigma b$ but $a \neq b$ we will write $a >_\sigma b$ or $b <_\sigma a$.

3.1.1. Definition. An *admissible ordering* $\sigma$ on $\langle X\rangle$ is a relation on $\langle X\rangle$ satisfying the following conditions for all $w_1, w_2, w_3, w_4 \in \langle X\rangle$:

(i) $\sigma$ is complete, i.e. $w_1 \geq_\sigma w_2$ or $w_2 \geq_\sigma w_1$.
(ii) $\sigma$ is reflexive, i.e. $w_1 \geq_\sigma w_1$.
(iii) $\sigma$ is antisymmetric, i.e. $w_1 \geq_\sigma w_2$ and $w_2 \geq_\sigma w_1$ implies $w_1 = w_2$.
(iv) $\sigma$ is transitive, i.e. $w_1 \geq_\sigma w_2$ and $w_2 \geq_\sigma w_3$ implies $w_1 \geq w_3$.
(v) $\sigma$ is compatible with multiplication, i.e. $w_1 \geq_\sigma w_2$ implies $w_3w_1w_4 \geq_\sigma w_3w_2w_4$.
(vi) $\sigma$ is a well-ordering, i.e. every descending chain of words $w_1 \geq_\sigma w_2 \geq_\sigma \dots$ in $\langle X\rangle$ becomes stationary eventually.

If $\sigma$ is an admissible ordering, then we have $w \geq_\sigma 1$ for 1 being the empty word and any word $w \in \langle X\rangle$.

3.1.2. Example.

- The *lexicographic ordering* on $\langle X\rangle$, denoted by `Lex` is defined as follows: for any two words $w_1, w_2 \in \langle X\rangle$, we say $w_1 \geq_{\texttt{Lex}} w_2$ if we have $w_1 = w_2w$ for any word $w \in \langle X\rangle$, or if we have $w_1 = wx_{i_1}w'$, $w_2 = wx_{i_2}w''$ for some words $w, w', w'' \in \langle X\rangle$ and some letters $x_{i_1}, x_{i_2} \in X$ such that $i_1 < i_2$.

  Note that the lexicographic ordering is not an admissible ordering, since it does not fulfill conditions (v) and (vi) of Definition 3.1.1. However, it is still useful as a tiebreaker when constructing admissible orderings.
- The *degree lexicographic ordering* on $\langle X\rangle$, denoted by `DegLex`, is defined as follows: for two words $w_1, w_2 \in \langle X\rangle$, we say $w_1 \geq_{\texttt{DegLex}} w_2$ if we have $\mathrm{len}(w_1) > \mathrm{len}(w_2)$ or if we have $\mathrm{len}(w_1) = \mathrm{len}(w_2)$ and $w_1 \geq_{\texttt{Lex}} w_2$.
- The *degree reverse lexicographic ordering* on $\langle X\rangle$, denoted by `DegRevLex`, is defined similarly: for two words $w_1, w_2 \in \langle X\rangle$, we say $w_1 \geq_{\texttt{DegLex}} w_2$ if we have $\mathrm{len}(w_1) > \mathrm{len}(w_2)$ or if we have $\mathrm{len}(w_1) = \mathrm{len}(w_2)$ and $w_1 \leq_{\texttt{Lex}} w_2$.

Both the degree lexicographic ordering and the degree reverse lexicographic ordering are admissible orderings.

From now on, we will assume that $\sigma$ is an admissible ordering on $\langle X\rangle$.

3.1.3. Definition. Let $f \in K\langle X\rangle \backslash \{0\}$ be a polynomial. Then $f$ can be uniquely represented as

$$f = c_1w_1 + \ldots + c_sw_s$$

with $c_1, \ldots, c_s \in K \setminus \{0\}$, $w_1, \ldots, w_s \in \langle X\rangle$ such that $w_1 >_\sigma \ldots >_\sigma w_s$. We call the set $\{w_1, \ldots, w_s\}$ the *support* of $f$, $\mathrm{Supp}(f)$. We call $\mathrm{LT}_\sigma(f) := w_1$ the *leading term* of $f$ with respect to $\sigma$ and $\mathrm{LC}_\sigma(f) := c_1$ the *leading coefficient* of $f$ with respect to $\sigma$. We also say $\mathrm{LM}_\sigma(f) := \mathrm{LC}_\sigma(f)\,\mathrm{LT}_\sigma(f)$ is the *leading monomial* of $f$ with respect to $\sigma$. If $\mathrm{LC}_\sigma(f) = 1$, we say $f$ is *monic*.

The leading term and leading coefficient $\mathrm{LT}_\sigma(0)$ and $\mathrm{LC}_\sigma(0)$ of the zero polynomial remain undefined.

3.1.4. Remark. Let $f, f_1, f_2$ be polynomials in $K\langle X\rangle \setminus \{0\}$.

(i) If $f_1 + f_2 \neq 0$ we have $\mathrm{LT}_\sigma(f_1 + f_2) \leq_\sigma \max\{\mathrm{LT}_\sigma(f_1), \mathrm{LT}_\sigma(f_2)\}$. Moreover, $\mathrm{LT}_\sigma(f_1 + f_2) = \max\{\mathrm{LT}_\sigma(f_1), \mathrm{LT}_\sigma(f_2)\}$ if and only if $\mathrm{LT}_\sigma(f_1) \neq \mathrm{LT}_\sigma(f_2)$ or $\mathrm{LC}_\sigma(f_1) + \mathrm{LC}_\sigma(f_2) \neq 0$.

(ii) For all words $w, w' \in K\langle X\rangle$, we have $\mathrm{LT}_\sigma(wfw') = w\,\mathrm{LT}_\sigma(f)w'$.

(iii) We have $\mathrm{LT}_\sigma(f_1 f_2) = \mathrm{LT}_\sigma(f_1)\,\mathrm{LT}_\sigma(f_2)$.

3.1.5. Definition. Let $I \subseteq K\langle X\rangle$ be an ideal.

(i) The (monomial) ideal $\mathrm{LT}_\sigma(I) \coloneqq \langle \mathrm{LT}_\sigma(f) |\, f \in I \setminus \{0\}\rangle \subseteq K\langle X\rangle$ is called the *leading term ideal* of $I$ with respect to $\sigma$.

(ii) The set $\mathrm{LT}_\sigma\{I\} \coloneqq \{\mathrm{LT}_\sigma(f) |\, f \in I \setminus \{0\}\} \subseteq K\langle X\rangle$ is called the *leading term set* of $I$ with respect to $\sigma$.

(iii) The set $\mathcal{O}_\sigma(I) \coloneqq \langle X\rangle \setminus \mathrm{LT}_\sigma\{I\}$ is called the *order ideal* of $I$ with respect to $\sigma$.

We define $\mathrm{LT}_\sigma(\langle 0\rangle) \coloneqq \langle 0\rangle$ and $\mathrm{LT}_\sigma\{0\} \coloneqq \varnothing$.

3.1.6. Lemma. *Let $I \subseteq K\langle X\rangle$ be an ideal.*

*(i) We have $K\langle X\rangle = I \oplus \mathrm{Span}_K\, \mathcal{O}_\sigma(I)$.*

*(ii) For every polynomial $f \in K\langle X\rangle$, there exists a unique polynomial $\hat{f} \in \mathrm{Span}_K\, \mathcal{O}_\sigma(I)$ such that $f - \hat{f} \in I$.*

3.1.7. Definition. Let $I \subseteq K\langle X\rangle$ be an ideal and $f \in K\langle X\rangle$ a polynomial. The unique polynomial $\hat{f} \in \mathrm{Span}_K\, \mathcal{O}_\sigma$ from Lemma 3.1.6 is called the *normal form* of $f$ modulo $I$ with respect to $\sigma$ and is denoted by $\mathrm{NF}_{\sigma,I}(f)$.

We say $f$ is *in normal form* modulo $I$ with respect to $\sigma$ if $f = \mathrm{NF}_{\sigma,I}(f)$.

3.1.8. Remark. We collect some properties of the normal form. Let $I \subseteq K\langle X\rangle$ be an ideal.

(i) For $f \in K\langle X\rangle$, we have $\mathrm{NF}_{\sigma,I}(\mathrm{NF}_{\sigma,I}(f)) = \mathrm{NF}_{\sigma,I}(f)$.

(ii) For $f_1, f_2 \in K\langle X\rangle$, we have $\mathrm{NF}_{\sigma,I}(f_1 - f_2) = \mathrm{NF}_{\sigma,I}(f_1) - \mathrm{NF}_{\sigma,I}(f_2)$.

(iii) For $f_1, f_2 \in K\langle X\rangle$, we have $\mathrm{NF}_{\sigma,I}(f_1) = \mathrm{NF}_{\sigma,I}(f_2)$ if and only if $f_1 - f_2 \in I$. In particular for any $f \in K\langle X\rangle$ we have $f \in I$ if and only if $\mathrm{NF}_{\sigma,I}(f) = 0$.

(iv) For $f_1, f_2 \in K\langle X\rangle$ we have $\mathrm{NF}_{\sigma,I}(f_1 f_2) = \mathrm{NF}_{\sigma,I}(\mathrm{NF}_{\sigma,I}(f_1)\,\mathrm{NF}_{\sigma,I}(f_2))$.

3.1.9. Remark. The uniqueness of the normal form allows us to solve the word problem in an algorithmic way. Suppose we have a ring $\mathcal{R} = \langle X |\, Rel\rangle$ that is defined by some finite set of relations $Rel \subseteq K\langle X\rangle \times K\langle X\rangle$ and let moreover $I \subseteq K\langle X\rangle$

be the ideal generated by the set $\{w - w' \mid (w, w') \in Rel\}$. Then two words $u$ and $v$ define the same element in $\mathcal{R}$ if and only if $u - v \in I$. By the previous remark, this is the case if and only if $\mathrm{NF}_{\sigma,I}(u - v) = 0$. Therefore computing the normal form can solve the word problem.

The algorithm that yields the normal form given a Gröbner basis is called the division algorithm. We do not give the algorithm here, but only mention the important result, the algorithm can be found e.g. in Theorem 3.2.1 in [93].

3.1.10. THEOREM. *There exists a* Division algorithm *which, given a polynomial $f \in K\langle X \rangle$ and a set of polynomials $G = \{g_1, \dots, g_s\} \subseteq K\langle X \rangle$ returns tuples $(c_{11}, w_{11}, w'_{11}), \dots, (c_{sk_s}, w_{sk_s}, w'_{sk_s})$ and a polynomial $p \in K\langle X \rangle$ such that*

$$f = \sum_{i=1}^{s} \sum_{j=1}^{k_i} c_{ij} w_{ij} g_i w'_{ij} + p$$

*and moreover:*

*(i) No element of* $\mathrm{Supp}(p)$ *is contained in* $\langle \mathrm{LT}_\sigma(g_1), \dots, \mathrm{LT}_\sigma(g_s) \rangle$.
*(ii) For all $i \in \{1, \dots, s\}$ and all $j \in \{1, \dots, k_i\}$, we have* $\mathrm{LT}_\sigma(w_{ij} g_i w'_{ij}) \leq_\sigma \mathrm{LT}_\sigma(f)$. *If $p \neq 0$ we have* $\mathrm{LT}_\sigma(p) \leq \mathrm{LT}_\sigma(f)$.
*(iii) For all $i \in \{1, \dots, s\}$ and all $j \in \{1, \dots, k_i\}$, we have* $\mathrm{LT}_\sigma(w_{ij} g_i w'_{ij}) \notin \langle \mathrm{LT}_\sigma(g_1), \dots, \mathrm{LT}_\sigma(g_{i-1}) \rangle$.

*We call* $\mathrm{NR}_{\sigma,G} \coloneqq p$ *the* normal remainder *of $f$ with respect to $\sigma$ and $G$.*

3.1.11. DEFINITION. Let $G \subseteq K\langle X \rangle \setminus \{0\}$ be a set of polynomials that generates an ideal $I = \langle G \rangle$. We say $G$ is a $(\sigma)$-*Gröbner basis* of $I$ if

$$\mathrm{LT}_\sigma\{I\} = \{w \, \mathrm{LT}_\sigma(g) w' \mid g \in G, w, w' \in \langle X \rangle\}.$$

3.1.12. PROPOSITION. *Let $G \subseteq K\langle X \rangle \setminus \{0\}$ be a $\sigma$-Gröbner basis that generates the ideal $I = \langle G \rangle$. Then we have that* $\mathrm{NR}_{\sigma,G}(f) = \mathrm{NF}_{\sigma,I}(f)$ *for all polynomials $f \in K\langle X \rangle$.*

3.1.13. REMARK. The above proposition is the reason why Gröbner bases in particular are such an important tool: together with Remark 3.1.9 we see that given a Gröbner basis for an ideal of relations, we can answer the word problem for the ring defined by these relations by simply executing the division algorithm.

Since we have now seen, why Gröbner bases are important, we will focus the rest of the chapter on giving an idea of how Gröbner bases can be computed. For this, let us fix a set of polynomials $G = \{g_1, \dots, g_s\}$, which defines an ideal $I = \langle G \rangle$ and which is supposed to be turned into a Gröbner basis.

We begin by introducing obstructions, which play an analogous role to critical syzygies in the commutative case.

3.1.14. Definition. Let $r \geq 1$. The $K\langle X\rangle$-bimodule $(K\langle X\rangle \otimes_K K\langle X\rangle)^r$, denoted by $F_r$, is called the *free bimodule* over $K\langle X\rangle$ of *rank* $r$. It has canonical basis $\{e_1, \dots, e_r\}$, i.e. $e_i = (0, \dots, 0, 1 \otimes 1, 0, \dots, 0)$ with the $1 \otimes 1$ being in the $i$-th position.

3.1.15. Definition. Let $i, j \in \{1, \dots, s\}$ and $i \leq j$. The element

$$o_{i,j}(w_i, w_i'; w_j, w_j') \coloneqq \frac{1}{\mathrm{LC}_\sigma(g_i)} w_i e_i w_i' - \frac{1}{\mathrm{LC}_\sigma(g_j)} w_j e_j w_j' \in F_s \setminus \{0\}$$

with $w_i, w_i', w_j, w_j' \in \langle X\rangle$ such that $w_i \operatorname{LT}_\sigma(g_i) w_i' = w_j \operatorname{LT}_\sigma(g_j) w_j'$ is called an *obstruction* of $g_i$ and $g_j$. If $i = j$ it is called a *self-obstruction* of $g_i$. The *set of all obstructions* of $g_i$ and $g_j$ is denoted by $o(i, j)$.

3.1.16. Definition.

- Let $w_1, w_2 \in \langle X\rangle$ be two words. If there exist $w, w', w'' \in \langle X\rangle$ with $w \neq 1$ such that $w_1 = w'w$ and $w_2 = ww''$ or $w_1 = w$ and $w_2 = w'ww''$, or vice versa, then we say $w_1$ and $w_2$ have an *overlap* at $w$. Otherwise we say $w_1$ and $w_2$ have *no overlap.*
- We say an obstruction $o_{i,j}(w_i, w_i'; w_j, w_j') \in o(i, j)$ has an *overlap* at $w \in \langle X\rangle \setminus \{1\}$ if $\operatorname{LT}_\sigma(g_i)$ and $\operatorname{LT}_\sigma(g_j)$ have an overlap at $w$ and if $w$ is a subword of $w_i \operatorname{LT}_\sigma(g_i) w_i'$. Otherwise we say $o_{i,j}(w_i, w_i'; w_j, w_j')$ has *no overlap.*
- If $i, j \in \{1, \dots, s\}$ and $i \leq j$ then we call an obstruction from $o(i, j)$ *non-trivial* if it has an overlap and is of one of the following forms:
  - $o_{i,j}(w_i, 1; 1, w_j')$
  - $o_{i,j}(1, w_i'; w_j, 1)$
  - $o_{i,j}(w_i, w_i'; 1, 1)$
  - $o_{i,j}(1, 1; w_j, w_j')$

  with $w_i, w_i', w_j, w_j' \in \langle X\rangle$.
- If $i \in \{1, \dots, s\}$, we say a self-obstruction in $o(i, i)$ is *non-trivial* if it has an overlap and is of the form $o_{i,i}(1, w_i'; w_i, 1)$ with $w_i, w_i' \in \langle X\rangle \setminus \{1\}$.
- Let $i, j \in \{1, \dots, s\}$ and $i \leq j$. We denote the *set of all non-trivial obstructions* of $g_i$ and $g_j$ by $O(i, j)$.

3.1.17. Definition. Let $i, j \in \{1, \dots, s\}$ with $i \leq j$ and let $o_{i,j}(w_i, w_i'; w_j, w_j') \in o(i, j)$ be an obstruction of $g_i$ and $g_j$. We call

$$S_{i,j}(w_i, w_i'; w_j, w_j') \coloneqq \frac{1}{\mathrm{LC}_\sigma(g_i)} w_i g_i w_i' - \frac{1}{\mathrm{LC}_\sigma(g_j)} w_j g_j w_j' \in K\langle X\rangle$$

the *S-polynomial* of $o_{i,j}(w_i, w_i'; w_j, w_j')$.

S-polynomials are important for computing Gröbner basis due to the following proposition.

3.1.18. Proposition (Buchberger Criterion). *The following conditions for $G$ are equivalent:*

*(i) The set $G$ is a $\sigma$-Gröbner basis of $I$.*

*(ii) For every obstruction $o_{i,j}(w_i, w_i'; w_j, w_j') \in \cup_{1 \le i \le j \le s} O(i,j)$, we have*

$$\mathrm{NR}_{\sigma,G}(S_{i,j}(w_i, w_i'; w_j, w_j')) = 0.$$

3.1.19. Theorem ((Buchberger procedure)). *Let $G \subseteq K\langle X\rangle$ be a finite list of polynomials which generate an ideal $I = \langle G\rangle$.*

*(i) For each $i, j \in \{1, \ldots, s\}$ with $i \le j$, we compute the set $O(i,j)$ of all nontrivial obstructions of $g_i$ and $g_j$. We initialize the set $B$ as the union of all obstruction sets $O(i,j)$ of $G$.*

*(ii) If $B = \varnothing$, return $G$. Otherwise select an obstruction $o_{i,j}(w_i, w_i'; w_j, w_j') \in B$ using a fair strategy and delete it from $B$. A selection strategy is fair if it ensures that every obstruction is eventually selected.*

*(iii) Compute the $S$-polynomial $S = S_{i,j}(w_i, w_i'; w_j, w_j')$ and its normal remainder $S' \coloneqq \mathrm{NR}_{\sigma,G}(S)$. If $S' = 0$, continue with step (ii).*

*(iv) Append $S'$ to $G$. Append the obstructions $O(i, s{+}1)$ for all $i \in \{1, \ldots, s{+}1\}$ to $B$. Increase $s$ by one. Continue with step (ii).*

*The above algorithm enumerates a $\sigma$-Gröbner basis $G$ of $I$. If $I$ has a finite $\sigma$-Gröbner basis, it will stop after finitely many steps and return a finite $\sigma$-Gröbner basis of $I$.*

## 3.2. Computing Gröbner Bases for Quantum Permutation Groups

Our motivation for using noncommutative Gröbner bases is to study quantum permutation groups. Since all our examples are given as universal $C^*$-algebras, one notices a problem in Definition 1.2.4 that can occur when handling these objects on a computer: computing the Gröbner basis of an ideal will only be able to handle any algebraic relation induced by the defining relations. However, in the second step of the construction of a universal $C^*$-algebra, we mod out the set $\{x \in A(E,R) | \ \|x\| = 0\}$, which can potentially add some analytical relations to our $C^*$-algebra. We will now argue why this will not be a problem in our case by recollecting some definitions and statements from [78].

The essential part is that when considering algebraic compact matrix quantum groups, the prenorm that one gets in the definition of the universal $C^*$-algebra is already a norm, and therefore the set $\{x \in A(E,R) | \ \|x\| = 0\}$ is just $\{0\}$. To get to these results, we first need to define *Hopf $*$-algebras* and algebraic compact quantum groups.

3.2.1. Definition. A *Hopf $*$-algebra* over $\mathbb{C}$ is a unital $*$-algebra $A$ that is also a Hopf algebra, i.e. that is equipped with a unital homomorphism that is called the *coproduct* $\Delta : A \to A \otimes A$, a homomorphism called the *counit* $\varepsilon : A \to \mathbb{C}$ and a linear map called the *antipode* $S : A \to A$ such that the following conditions hold:

(i) $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$

(ii) $(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta$
(iii) $m \circ (S \otimes \mathrm{id}_A) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id}_A \otimes S) \circ \Delta$

where $m : A \otimes A \to A$, $a \otimes b \mapsto ab$ denotes the multiplication map and $\eta : \mathbb{C} \to A$, $\lambda \mapsto \lambda \mathbf{1}_A$ denotes the unit map.

An *algebraic compact quantum group* is a Hopf $*$-algebra with a positive integral. The unique normalized positive integral on an algebraic compact quantum group is called its *Haar state* and denoted by $h$.

Analogous to ($C^*$-algebraic) compact matrix quantum groups, there are also algebraic compact matrix quantum groups.

3.2.2. Definition. An *algebraic compact matrix quantum group* is a Hopf $*$-algebra $(A, \Delta)$ together with a unitary $u \in M_n(A)$, where $n \in \mathbb{N}$, such that

(i) $u$ is a corepresentation matrix.
(ii) $\overline{u}$ is equivalent to a unitary corepresentation matrix.
(iii) The elements $u_{ij}$ for $i, j \in \{1, \ldots, \mathbb{N}\}$ generate $A$ as a $*$-algebra.

In the above case, the counit and the antipode of the Hopf $*$-algebra $(A, \Delta)$ are uniquely determined by $\varepsilon(u_{ij}) = \delta_{ij}$ and $S(u_{ij}) = u_{ij}^*$ for all $i, j$.

The name compact matrix quantum group in the above definition is justified by the following proposition.

3.2.3. Proposition (Prop. 6.1.4 (i) in[78]). *If $(A, \Delta, u)$ is an algebraic compact matrix quantum group, then $(A, \Delta)$ is an algebraic compact quantum group and $u$, $\overline{u}$ are corepresentation matrices of $(A, \Delta)$.*

The last piece we need is the following statement, which is part of Theorem 5.4.3 in [78].

3.2.4. Proposition. *Let $(A, \Delta)$ be an algebraic compact quantum group. Then*

$$a \mapsto \|a\|_u \coloneqq \sup\{p(a) \mid p \text{ is a } C^*\text{-seminorm on } A\}$$

*is a $C^*$-norm on $A$.*

We see that the seminorm $\|\cdot\|_u$ is exactly the same as the one that appears in Definition 1.2.4. Since by the above proposition it is already a norm, the set $\{x| \, \|x\|_u = 0\}$ is trivial and therefore we do not get any additional relations from this construction.

3.2.5. Proposition. *Let $(C(G), u)$ be a quantum permutation group and let $(E, R)$ be a finite set of variables and a set of relations in those variables such that $C(G) = C^*(E|R)$. Let moreover $I(R)$ be the ideal generated by $R$ in the free $*$-algebra $P(E)$ and let $A(E, R) \coloneqq P(E)/I(R)$.*

*Then for any element $x$ of $A(E, R)$ we have:*

$$x = 0 \text{ in } A(E, R) \iff x = 0 \text{ in } C(G).$$

Proof. Since by Proposition 3.2.4 the norm $\|\cdot\|_u$ is already a norm on $A(E,R)$ and since this is also the $C^*$-norm on $C^*(E|R) = C(G)$, the claim follows. $\square$

3.2.6. Remark. The above proposition is important since it tells us that if we find a Gröbner basis of the defining ideal of a quantum permutation group then it can already completely decide equality of any polynomials in the corresponding $C^*$-algebra, and in particular by choosing a norm and doing a closure in that norm, we do not get any extra relations.

## 3.3. Applications in this Thesis

In Section 4.5, we use noncommutative Gröbner basis computations to find the quantum automorphism groups of two graphs. We use the Gröbner bases to compute certain relations on the generators, that allow us to construct the $*$-isomorphisms. For the computations, we use the implementation in `OSCAR`, which was co-written by the author of this thesis.

3.3.1. Theorem (4.5.1, 4.5.2). *The graphs* $C_{12}(4,5)$ *and* $C_{12}(3^+,6)$ *have the quantum automorphism group* $H_2^+ \times S_3$.

For the proofs of the theorems mentioned above, we need to construct a $*$-isomorphism between the quantum automorphism group of the graph in question and the quantum group $H_2^+ \times S_3$. In order to do this, we need to check that several relations hold for the generators of the quantum automorphism group. Some of these were straightforward to check by hand, for others however we used the Gröbner basis, namely the relations given in Equations 4.5.1 and 4.5.2 and the relation that a few certain sums over some elements in the quantum automorphism group are 1. We see that Gröbner basis computations can be helpful when determining what exactly a given quantum automorphism group is, however there is still a lot of work to do by hand, since we need to know to which already known quantum group we want to construct a $*$-isomorphism.

This is a bit different when one is only interested in the question, whether or not a given quantum automorphism group is the same as the classical automorphism group. For this, one can compute the Gröbner basis of the defining $C^*$-algebra and then simply check whether the commutation relations of all the generators hold. This is what we did for a number of matroids in Section 6.5.

CHAPTER 4

# Quantum Symmetries of Vertex-Transitive Graphs on 12 Vertices

In this chapter, which is based on the article [73] by the author, we consider the quantum symmetries of vertex-transitive graphs on 12 vertices. Vertex-transitive graphs are interesting because the vertex-transitivity ensures that the graphs have at least a certain amount of symmetry. In [6], Banica and Bichon described the quantum automorphism goups of all vertex-transitive graphs on up to 11 vertices except for the Petersen graph, for which Schmidt showed in [74] that it does not have quantum symmetries. Moreover, Chassaniol described the quantum automorphism groups of all vertex-transitive graphs on 13 vertices in [22], [23]. This section fills the gap by investigating all vertex-transitive graphs on 12 vertices to see which of these have quantum symmetries and which do not, partially answering a question posed in the PhD thesis of Schmidt [75].

We sort the vertex-transitive graphs on 12 vertices in 5 subclasses: disconnected graphs, products of smaller graphs, circulant graphs, semicirculant graphs and special cases that do not fit into any of the other subclasses.

In total, there are 74 vertex-transitive graphs on 12 vertices. However, since the quantum automorphism group of a graph is the same as the one of its complement, one only needs to consider one of the two. Due to this, we study only 37 graphs.

The main result of this section is that we determine the existence of quantum symmetries of vertex-transitive graphs on 12 vertices. Combining this with previous results of Banica and Bichon in [6], Schmidt in [74] and Chassaniol in [22], [23], we get the following:

4.1. Theorem. *For vertex transitive graphs on up to* 13 *vertices, the existence of quantum symmetries is completely determined.*

In the following table, we give a first overview over the graphs we studied.

| subclass | total graphs | graphs with quantum symmetries |
|---|---|---|
| disconnected graphs | 9 | 9 |
| products of smaller graphs | 6 | 5 |
| circulant graphs | 12 | 3 |
| semi-circulant graphs | 5 | 3 |
| special cases | 5 | 0 |

As one can see, 20 of the 37 graphs we studied have quantum symmetries. Among those with quantum symmetries there are all graphs that are disjoint copies of smaller graphs, some graph products, three circulant and three semi-circulant graphs.

In the following table, we recollect the previous results on quantum symmetries of vertex-transitive graphs by Banica and Bichon in [6] and by Chassaniol in [22], [23]. Also included is the result that the Petersen graph has no quantum symmetries by Schmidt [74]. These are all vertex transitive graphs of the given order up to complements. We exclude the trivial cases of the full graphs and of disjoint copies of smaller graphs, due to the fact that the full graph on $n$ vertices always has quantum automorphism group $S_n^+$, which can be seen easily, and due to the result of Banica and Bichon in [5], that a graph consisting of $n$ disjoint copies of a connected graph $\Gamma$ has quantum automorphism group

$$G^+_{aut}(n\Gamma) = G^+_{aut}(\Gamma) \wr_* S_n^+.$$

TABLE 1. Vertex-transitive graphs on $\leq 11$ and on 13 vertices

| Order | Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|---|
| $n \neq 4$ | $C_n$ | $D_n$ | $D_n$ |
| 8 | $C_8(4)$ | $D_8$ | $D_8$ |
| 8 | $K_2 \square C_4$ | $S_4 \times \mathbb{Z}_2$ | $S_4^+ \times \mathbb{Z}_2$ |
| 9 | $C_9(3)$ | $D_9$ | $D_9$ |
| 9 | $K_3 \square K_3$ | $S_3 \wr \mathbb{Z}_2$ | $S_3 \wr \mathbb{Z}_2$ |
| 10 | $C_{10}(2)$ | $D_{10}$ | $D_{10}$ |
| 10 | $C_{10}(5)$ | $D_{10}$ | $D_{10}$ |
| 10 | $K_2 \square C_5$ | $D_{10}$ | $D_{10}$ |
| 10 | Petersen | $S_5$ | $S_5$ |
| 10 | $K_2 \square K_5$ | $S_5 \times \mathbb{Z}_2$ | $S_5^+ \times \mathbb{Z}_2$ |
| 10 | $C_{10}(4)$ | $\mathbb{Z}_2 \wr D_5$ | $\mathbb{Z}_2 \wr_* D_5$ |
| 11 | $C_{11}(2)$ | $D_{11}$ | $D_{11}$ |
| 11 | $C_{11}(3)$ | $D_{11}$ | $D_{11}$ |
| 13 | $C_{13}$ | $D_{13}$ | $D_{13}$ |
| 13 | $C_{13}(2)$ | $D_{13}$ | $D_{13}$ |
| 13 | $C_{13}(2,5)$ | $D_{13}$ | $D_{13}$ |
| 13 | $C_{13}(2,6)$ | $D_{13}$ | $D_{13}$ |
| 13 | $C_{13}(3)$ | $D_{13}$ | $D_{13}$ |
| 13 | $C_{13}(5)$ | $\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$ | $\mathbb{Z}_{13} \rtimes \mathbb{Z}_4$ |
| 13 | $C_{13}(3,4)$ | $\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$ | $\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$ |

Next, we give a table of all vertex-transitive graphs on 12 vertices up to complements with their automorphism groups and their quantum automorphism groups, if it is known. For the sake of completeness, we include the trivial cases in this table. In particular, we have computed for all of the following graphs, whether they have quantum symmetries. Therefore, whenever the column with the quantum automorphism group is a question mark, the graph in question does have quantum symmetries, but we did not manage to find the actual quantum automorphism group.

TABLE 2. Vertex-transitive graphs on 12 vertices

| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $6K_2$ | $\mathbb{Z}_2 \wr S_6$ | $\mathbb{Z}_2 \wr_* S_6^+$ |
| $4K_3$ | $S_3 \wr S_4$ | $S_3 \wr_* S_4^+$ |
| $3K_4$ | $S_4 \wr S_3$ | $S_4^+ \wr_* S_3$ |
| $3C_4$ | $H_2 \wr S_3$ | $H_2^+ \wr_* S_3$ |
| $2K_6$ | $S_6 \wr \mathbb{Z}_2$ | $S_6^+ \wr_* \mathbb{Z}_2$ |
| $2C_6$ | $D_6 \wr \mathbb{Z}_2$ | $D_6 \wr_* \mathbb{Z}_2$ |
| $2(K_2 \Box K_3)$ | $D_6 \wr \mathbb{Z}_2$ | $D_6 \wr_* \mathbb{Z}_2$ |
| $2C_6(2)$ | $(\mathbb{Z}_2 \wr S_3) \wr \mathbb{Z}_2$ | $(\mathbb{Z}_2 \wr_* S_3) \wr_* \mathbb{Z}_2$ |
| $2C_6(3)$ | $(S_3 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$ | $(S_3 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$ |
| $K_6 \times K_2$ | $S_6 \times \mathbb{Z}_2$ | $S_6^+ \times \mathbb{Z}_2$ |
| $K_3 \times K_4$ | $S_3 \times S_4$ | $S_3 \times S_4^+$ |
| $C_4 \Box C_3$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $K_2 \Box C_6(3)$ | $\mathbb{Z}_2 \times (S_3 \wr \mathbb{Z}_2)$ | $\mathbb{Z}_2 \times (S_3 \wr_* \mathbb{Z}_2)$ |
| $K_2 \Box C_6$ | $\mathbb{Z}_2 \times D_6$ | $\mathbb{Z}_2 \times D_6$ |
| $K_2 \Box C_6(2)$ | $\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr S_3)$ | ? |
| $C_{12}$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(3)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(6)$ | $D_{12}$ | $D_{12}$ |
| $K_{12}$ | $S_{12}$ | $S_{12}^+$ |
| $C_{12}(5)$ | $\mathcal{A}$ | ? |
| $C_{12}(4,5)$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $C_{12}(5,6)$ | $\mathcal{A}$ | ? |
| $C_{12}(2)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(4)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(2,6)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(3,6)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(4,6)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(5^+)$ | $H_3$ | ? |
| $C_{12}(3^+,6)$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $C_{12}(5^+,6)$ | $H_3$ | ? |
| $C_{12}(2,5^+)$ | $D_6$ | $D_6$ |
| $C_{12}(4,5^+)$ | $D_6$ | $D_6$ |
| Icosahedral Graph | $\mathbb{Z}_2 \times A_5$ | $\mathbb{Z}_2 \times A_5$ |
| $L(C_6(2))$ | $H_3$ | $H_3$ |
| $Trunc(K_4)$ | $S_4$ | $S_4$ |
| $Antip(Trunc(K_4))$ | $S_4$ | $S_4$ |
| Cuboctahedral Graph | $H_3$ | $H_3$ |

Here, we call

$$\mathcal{A} := (\mathbb{Z}_2 \times ((((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2.$$

Looking at the data in the above tables, one can make some observations.

1. Observation. *For vertex transitive graphs on up to 13 vertices, the existence of quantum symmetries is equivalent to the existence of disjoint automorphisms.*

It is already known that the statement from Observation 1 does not hold for all graphs, as has been shown in [75], but it is an open question for what families of graphs this is an alternative characterisation of the existence of quantum symmetries.

We moreover observe the following fact.

2. Observation. *Let $\Gamma$ be a vertex-transitive graph on up to 13 vertices. If the automorphism group of $\Gamma$ is $D_n$ with $n \neq 4$, then $\Gamma$ does not have quantum symmetries.*

In addition to computing whether the graphs have quantum symmetries and giving the explicit quantum automorphism group wherever it was known by some previous result, we also compute the quantum automorphism group of two graphs for which it was not yet known in Section 4.5.

The data from the above tables was used in [16] to get an overview of vertex-transitive graphs with known quantum automorphism group in order to investigate them for the existence of so-called quantum twin vertices. In their paper, the authors introduce the concept of quantum twin vertices of quantum graphs as a generalisation of twin vertices of classical graphs and then ask the question whether quantum twin vertices of classical graphs are a broader concept than classical twin vertices of classical graphs. In particular they ask whether the graphs $K_2 \square K_5$ and $C_4 \square C_3$ possess quantum twin vertices, since these are the only graphs from the above tables that have quantum symmetries, have no twin vertices but vanishing determinant of the adjacency matrix.

## 4.1. Overview and Notation

On twelve vertices, there are in total 74 vertex-transitive graphs. Since the quantum automorphism group of a graph is the same as the one of its complement, it suffices to only consider 37 of these graphs, since the rest can be obtained by taking their complements.

We will sort these graphs in five different subclasses: disconnected graphs, products of smaller graphs, circulant graphs, semicirculant graphs and special cases that do not fit into any of the other subclasses.

In total, we will consider the following graphs:

| Subclass | Graphs |
|---|---|
| disconnected | $6K_2$, $4K_3$, $3K_4$, $3C_4$, $2K_6$, $2C_6$, |
| | $2K_2\square K_3$, $2C_6(2)$, $2C_6(3)$ |
| products | $K_6 \times K_2$, $K_3 \times K_4$, $C_6\square K_2$, |
| | $C_4\square C_3$, $K_2\square C_6(2)$, $K_2\square C_6^+$ |
| circulant | $C_{12}$, $K_{12}$, $C_{12}^+$, $C_{12}(2)$, $C_{12}(3)$, $C_{12}(4)$, |
| | $C_{12}(5)$, $C_{12}(2,6)$, $C_{12}(4,6)$, $C_{12}(3,6)$, |
| | $C_{12}(4,5)$, $C_{12}(5,6)$ |
| semi-circulant | $C_{12}(5^+)$, $C_{12}(3^+,6)$, $C_{12}(5^+,6)$, |
| | $C_{12}(2,5^+)$, $C_{12}(4,5^+)$ |
| special cases | L(Cube), L($C_6(2)$), Icosahedron, |
| | Trunc($K_4$), Antip(Trunc($K_4$)) |

Table 3. Vertex transitive graphs on 12 vertices up to complements

In the proofs that a graph has no quantum symmetries, we will often repeatedly apply Lemma 2.2.4 and Lemma 2.2.5. In order to prevent this from taking up too much space, we will now introduce the following notation:

4.1.1. Notation. Let $\Gamma$ be a graph and $j$ a fixed vertex in $\Gamma$. If we want to apply Lemma 2.2.4 to $j$ and $l_1, l_2, \dots$ and $q_1, q_2, \dots$ to get that $u_{ij}u_{kl_a} = u_{ij}u_{kl_a}\sum_{p\in P_a} u_{ip}$, we will write the following table:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| $j$ | $l_1$ | $q_1$ | $P_1$ |
| | $l_2$ | $q_2$ | $P_2$ |
| | $\vdots$ | $\vdots$ | $\vdots$ |

Here, the sets $P_a$ will be the sets of vertices $p$ such that $d(l_a, p) = d(j, l_a)$ and $d(p, q_a) = d(j, q_a)$. Often, we will use this when the sets $P_a$ are singleton sets, because this will mean that we have $u_{ij}u_{kl_a} = u_{ij}u_{kl_a}u_{ij}$.

4.1.2. Notation. Let $\Gamma$ be a graph and $j$ a fixed vertex in $\Gamma$. If we want to apply Lemma 2.2.5 to $j$, $l$, $p$ and $q$, we often want to apply it in such a way, that we get $u_{ij}u_{kl}u_{ip} = 0$. This is the case if $l$ is the unique vertex in distance $d(l,q)$ from $q$, $d(l,j)$ from $j$ and $d(l,p)$ from $p$. If that is the case, we will write the application of Lemma 2.2.5 to $j$, $l$, $p_1, p_2, \dots$ and $q_1, q_2, \dots$ as the following table:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| $j$ | $l$ | $p_1$ | $q_1$ |
| | | $p_2$ | $q_2$ |
| | | $\vdots$ | $\vdots$ |

A particular row of such a table containing $j', l', p', q'$ then means that $u_{ij'}u_{kl'}u_{ip'} = 0$ for all vertices $i, k \in V(\Gamma)$.

To see how an application of the newly introduced notations might look in practice and how one might start working on a new graph in general, we give a simple example:

4.1.3. Example. We consider the 5-cycle $C_5$. We already know, that it has no quantum symmetries, but we now want to show this using Lemma 2.2.5 and Lemma 2.2.4. By Corollary 2.2.8, we only need to show that the generators $u_{i1}$ commute with all generators $u_{kl}$. Note moreover, that by Lemma 2.2.1, we only need to show that $u_{i1}u_{kl} = u_{i1}u_{kl}u_{i1}$. We do this by using the fact from the relations of $G^+_{aut}(C_5)$ that $\sum_r u_{ir} = 1$ and thus

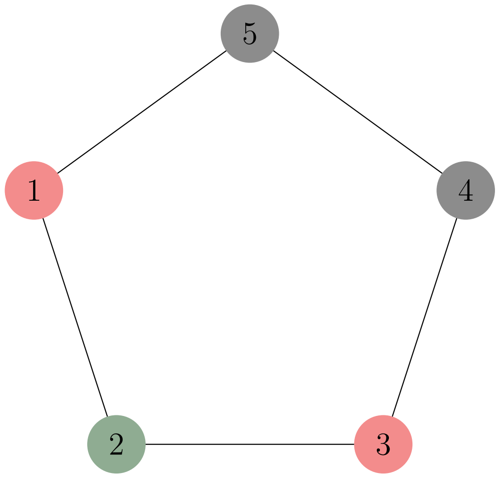
$$u_{i1}u_{kl} = u_{i1}u_{kl}\sum_p u_{ip}.$$

Using additionally Lemma 2.2.3, we get that

$$u_{i1}u_{kl}\sum_p u_{ip} = u_{i1}u_{kl}\sum_{\substack{p \\ d(p,l)=d(i,k)}} u_{ip}.$$

Now we just need to show that $u_{i1}u_{kl}u_{ip} = 0$ for any vertex $l$ and any $p \neq 1$ with $d(p,l) = d(i,k)$. Here we can assume that $d(i,k) = d(1,l)$ without loss of generality, since whenever $d(i,k) \neq d(1,l)$, we have $u_{i1}u_{kl} = u_{kl}u_{i1} = 0$ and we are already done. Note, that in the case of $C_5$ for any vertex $l$, only one vertex $p \neq 1$ is in the same distance to $l$ as 1 and therefore the above means that for any $l$, we only have to show $u_{i1}u_{kl}u_{ip} = 0$ for a single vertex $p$.

In order to show that $u_{i1}u_{kl}u_{ip} = 0$ for the above specified vertices $l$ and $p$, we will first use Lemma 2.2.5. To get that $u_{i1}u_{k2} = u_{k2}u_{i1}$, we will now use the lemma to show that $u_{i1}u_{k2}u_{i3} = 0$. Looking at the conditions for applying Lemma 2.2.5, we thus want to find a vertex $q$ such that $d(1, q) \neq d(q, 3)$ and such that 2 is the only vertex that is both in distance $s = d(2, q)$ from $q$ and in distance 1 from both vertices 1 and 3. Choosing $q := 1$ satisfies this as can be seen by looking at the graph:




We can therefore apply Lemma 2.2.5 to get that $u_{i1}u_{k2}u_{i3} = 0$ and therefore

$$u_{i1}u_{k2} = u_{i1}u_{k2}u_{i1}$$

as desired. In a very similar manner, we can apply Lemma 2.2.5 to $l = 5$. Both of these applications are summarised in the following table, using the notation from 4.1.2:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 1 |
| | 5 | 4 | 1 |

This table thus already shows that the $u_{i1}$ commute with all $u_{kl}$ where $l$ is adjacent to 1. Using Lemma 2.2.7, we thus get that all $u_{ij}$ and $u_{kl}$ commute whenever $j \sim l$. We now want to use this information to apply Lemma 2.2.4 to show that $u_{i1}u_{k3} = u_{i1}u_{k3}u_{i1}$. We thus have to find a vertex $q$ with $d(q, 3) = 1$ such that there is exactly one vertex $p$ that satisfies $d(3, p) = 2$ and $d(p, q) = d(j, q)$. Choosing $q := 2$ only leaves the vertex $p = 1$ satisfying these conditions and therefore Lemma 2.2.4 yields

$$u_{i1}u_{k3} = u_{i1}u_{k3}u_{i1}$$

and therefore also

$$u_{i1}u_{k3} = u_{k3}u_{i1}.$$

In a similar manner, we can apply Lemma 2.2.4 to get $u_{i1}u_{k4} = u_{i1}u_{k4}u_{i1}$, and again, both of these applications are summarised in the following table according to Notation 4.1.1:

Lemma 2.2.5

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 3 | 2 | $\{1\}$ |
| | 4 | 3 | $\{1\}$ |

All in all, we see that $u_{i1}u_{kl} = u_{kl}u_{i1}$ for all vertices $i, k$ and $l$ and thus by Lemma 2.2.7 we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all generators of the quantum automorphism group of $C_5$.

## 4.2. Constructions from Smaller Graphs

In this section we consider graphs that arise as constructions of smaller graphs. A lot of the time, their quantum automorphism groups can be computed in a simple manner from the quantum automorphism groups of the smaller graphs.

**4.2.1. Disconnected graphs.** Vertex transitive graphs that are disconnected are always disjoint copies of a single connected graph. For such disjoint copies of a connected graph, it was shown in [5] how to compute the quantum automorphism groups:

4.2.1.1. THEOREM ([5]). *If $\Gamma$ is a connected graph, then the graph consisting of $n$ disjoint copies of $\Gamma$ has quantum automorphism group*

$$G^+_{aut}(n\Gamma) = G^+_{aut}(\Gamma) \wr_* S^+_n.$$

We thus can compute the quantum automorphism groups of these graphs easily.

| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $6K_2$ | $\mathbb{Z}_2 \wr S_6$ | $\mathbb{Z}_2 \wr_* S_6^+$ |
| $4K_3$ | $S_3 \wr S_4$ | $S_3 \wr_* S_4^+$ |
| $3K_4$ | $S_4 \wr S_3$ | $S_4^+ \wr_* S_3$ |
| $3C_4$ | $H_2 \wr S_3$ | $H_2^+ \wr_* S_3$ |
| $2K_6$ | $S_6 \wr \mathbb{Z}_2$ | $S_6^+ \wr_* \mathbb{Z}_2$ |
| $2C_6$ | $D_6 \wr \mathbb{Z}_2$ | $D_6 \wr_* \mathbb{Z}_2$ |
| $2(K_2 \square K_3)$ | $D_6 \wr \mathbb{Z}_2$ | $D_6 \wr_* \mathbb{Z}_2$ |
| $2C_6(2)$ | $(\mathbb{Z}_2 \wr S_3) \wr \mathbb{Z}_2$ | $(\mathbb{Z}_2 \wr_* S_3) \wr_* \mathbb{Z}_2$ |
| $2C_6(3)$ | $(S_3 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$ | $(S_3 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$ |

**4.2.2. Products of smaller graphs.** For finite graphs one can define the following product operations:

4.2.2.1. Definition. Let $\Gamma$ and $\Gamma'$ be finite graphs.

(i) The direct product $\Gamma \times \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and the following edges:

$$(i,k) \sim (j,l) \iff i \sim j \text{ and } k \sim l.$$

(ii) The Cartesian product $\Gamma \square \Gamma'$ has vertex set $V(\Gamma) \times V(\Gamma')$ and the following edges:

$$(i,k) \sim (j,l) \iff i = j, k \sim l \text{ or } i \sim j, k = l.$$

In [4], the following theorem about quantum automorphism groups of graph products was proven.

4.2.2.2. Theorem. *Let $\Gamma$ and $\Gamma'$ be finite connected regular graphs. If their spectra $\{\lambda\}$ and $\{\mu\}$ do not contain $0$ and satisfy*

$$\{\lambda_i/\lambda_j\} \cap \{\mu_k/\mu_l\} = \{1\},$$

*then $C(G^+_{aut}(\Gamma \times \Gamma')) = C(G^+_{aut}(\Gamma)) \otimes C(G^+_{aut}(\Gamma'))$.*

*If the spectra satisfy*

$$\{\lambda_i - \lambda_j\} \cap \{\mu_k - \mu_l\} = \{0\},$$

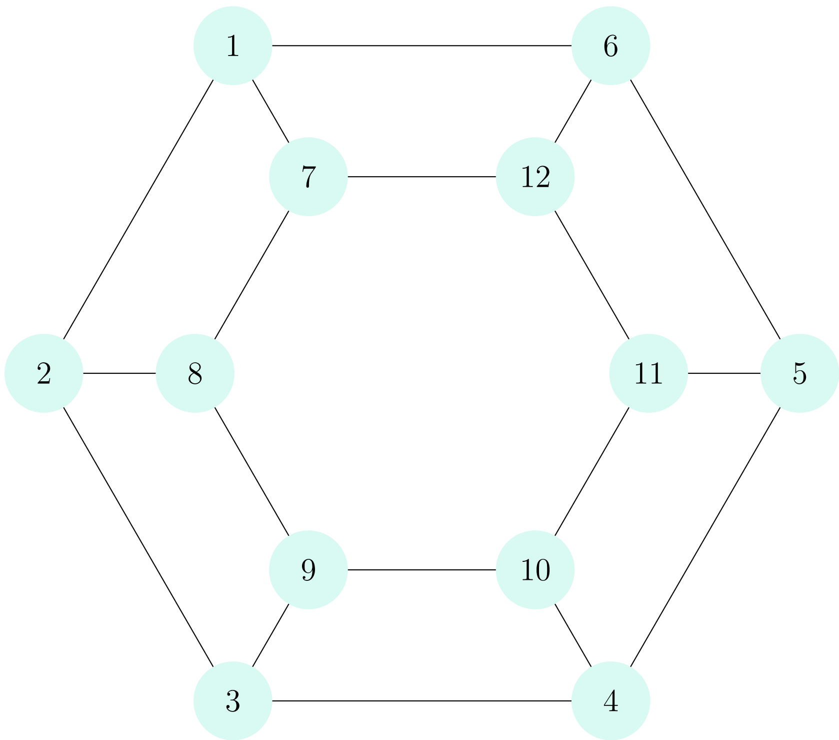
*then $C(G^+_{aut}(\Gamma \square \Gamma')) = C(G^+_{aut}(\Gamma)) \otimes C(G^+_{aut}(\Gamma'))$.*

This theorem yields the quantum automorphism group for a few of the product graphs already.

| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $K_6 \times K_2$ | $S_6 \times \mathbb{Z}_2$ | $S_6^+ \times \mathbb{Z}_2$ |
| $K_3 \times K_4$ | $S_3 \times S_4$ | $S_3 \times S_4^+$ |
| $C_4 \square C_3$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $K_2 \square C_6(3)$ | $\mathbb{Z}_2 \times (S_3 \wr \mathbb{Z}_2)$ | $\mathbb{Z}_2 \times (S_3 \wr_* \mathbb{Z}_2)$ |

For the two product graphs $K_2 \square C_6$ and $K_2 \square C_6(2)$, the theorem does not apply, and they therefore have to be studied individually. We will first consider$K_2 \square C_6$:

4.2.2.3. Proposition. *The graph $K_2 \square C_6$, pictured below, does not have quantum symmetry.*




Proof. We will show that $u_{i1}u_{kl} = u_{kl}u_{i1}$ for all vertices $i, k, l$ of $K_2 \square C_6$. Then the vertex transitivity will imply the commutativity of all generators.

We begin with distance 2. There are 4 vertices in distance 2 from 1: $3, 5, 8$ and 12. First considering vertex 3, we see that

$$u_{i1}u_{k3} = u_{i1}u_{k3}(u_{i1} + u_{i5} + u_{i8} + u_{i10}).$$

We will now apply Lemma 2.2.5 to 1 and 3 using the notation introduced in 4.1.2:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 3 | 5 | 2 |
| | | 8 | 6 |

We thus get that $u_{i1}u_{k3}u_{i5} = 0 = u_{i1}u_{k3}u_{i8}$. Recall now, that by Definition 2.2.12, we denote by $\mathrm{CN}(i, j)$ the set of common neighbours of the vertices $i$ and $j$. We can now see that $|\mathrm{CN}(1, 3)| = 1$, but we have $|\mathrm{CN}(3, 10)| = 2$, and we thus get with Corollary 2.2.14 that $u_{i1}u_{k3}u_{i10} = 0$. We thus get $u_{i1}u_{k3} = u_{k3}u_{i1}$.

Next, we consider vertex 8 and apply Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 8 | 3 | 4 |
| | | 12 | 2 |

Again, only $u_{i1}u_{k8}u_{i10}$ is the missing term, but seeing that $|\mathrm{CN}(1, 8)| = 2 \neq |\mathrm{CN}(8, 10) = 3$ we get again by Corollary 2.2.14 that $u_{i1}u_{k8}u_{i10} = 0$ and thus $u_{i1}u_{k8} = u_{k8}u_{i1}$.

We observe, that mirroring on the line that goes through $1, 7, 10$ and 4 is an automorphism of $K_2 \square C_6$ that keeps 1 fixed and maps 3 to 5 and 8 to 12, we get by Lemma 2.2.7 that $u_{i1}u_{k5} = u_{k5}u_{i1}$ and also $u_{i1}u_{k12} = u_{k12}u_{i1}$. We thus have for all vertices $j, l$ with $d(j, l) = 2$ that $u_{ij}u_{kl} = u_{kl}u_{ij}$.

For distance 1, we apply Lemma 2.2.4 to all neighbours of the vertex 1:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 2 | 6 | $\{1\}$ |
| | 6 | 2 | $\{1\}$ |
| | 7 | 2 | $\{1, 8\}$ |

We see that $u_{i1}u_{k2} = u_{k2}u_{i1}$ and $u_{i1}u_{k6} = u_{k6}u_{i1}$. For $l = 7$, we moreover see that

$$u_{i1}u_{k7} = u_{i1}u_{k7}\left(u_{i1} + u_{i8}\right).$$

Looking at $u_{i1}u_{k7}u_{i8}$, we see that

$$\begin{aligned} u_{i1}u_{k7}u_{i8} &= u_{i1}u_{k7} \sum_{\substack{r;d(r,i)=1 \\ d(r,k)=2}} u_{r9}u_{i8} \\ &= u_{i1} \sum_{\substack{r;d(r,i)=1 \\ d(r,k)=2}} u_{r9}u_{k7}u_{i8} && \text{since } d(7,9)=2 \\ &= 0 && \text{since } d(1,9)=3 \neq d(r,i)=1. \end{aligned}$$

For distance 3, it is enough to only apply Lemma 2.2.4:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 4 | 10 | $\{1\}$ |
| | 9 | 10 | $\{1\}$ |
| | 11 | 10 | $\{1\}$ |

Lastly, since 10 is the only vertex in distance 4 to 1, and thus also vice versa, we get

$$u_{i1}u_{k10} = u_{i1}u_{k10} \sum_{r;d(r,10)=4} u_{ir} = u_{i1}u_{k10}u_{i1}.$$

From this we get $u_{i1}u_{k10} = u_{k10}u_{i1}$ and therefore $C(G^+_{aut}(K_2 \square C_6))$ is commutative. $\square$

4.2.2.4. Proposition. *The graph $K_2 \square C_6(2)$ has quantum symmetries.*

Proof. We have the following automorphisms of $K_2 \square C_6(2)$:

$$([1,1][1,4])([2,1][2,4]) \in \mathrm{Aut}(K_2 \square C_6(2))$$
$$([1,2][1,5])([2,2][2,5]) \in \mathrm{Aut}(K_2 \square C_6(2)).$$

These are non-trivial and disjoint, and thus, by Lemma 2.2.2, $K_2 \square C_6(2)$ has quantum symmetries. $\square$

| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $K_2 \square C_6$ | $\mathbb{Z}_2 \times D_6$ | $\mathbb{Z}_2 \times D_6$ |
| $K_2 \square C_6(2)$ | $\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr S_3)$ | ? |

## 4.3. Circulant and Semi-circulant Graphs

Two related families of vertex-transitive graphs are circulant and semi-circulant graphs, which were defined in Definition 1.1.6 Roughly speaking they arise when taking a cycle-graph and adding edges between certain vertices. Prior results yield the non-existence of quantum symmetries for a few of these graphs, however most of them have to be considered individually.

**4.3.1. Circulant graphs.** For some circulant graphs, Theorem 3.1 in [6] shows that they do not have quantum symmetries.

4.3.1.1. THEOREM ([6]). *Given a circulant graph $C_n(k_1,\dots,k_r)$ and putting $k_0 := 1$, we define the function $f:\{1,2,\dots,[\frac{n}{2}]\}\to\mathbb{R}$ as follows:*

$$f(s)=\sum_{i=0}^{r}\cos\left(\frac{2k_i s\pi}{n}\right).$$

*If $n\neq 4$ and the function $f$ is injective, then the graph $C_n(k_1,\dots,k_r)$ does not have quantum symmetries.*

4.3.1.2. PROPOSITION. *The graphs $C_{12}$, $C_{12}(3)$ and $C_{12}(6)$ do not have quantum symmetries.*

PROOF. We apply Theorem 4.3.1.1, and thus need to compute the values of the associated function $f$ on the set $\{1,\dots,6\}$:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $C_{12}$ | 0.87 | 0.5 | 0 | −0.5 | −0.87 | −1 |
| $C_{12}(3)$ | 0.87 | −0.5 | −1.23 | 0.5 | −0.87 | −2 |
| $C_{12}(6)$ | −0.13 | 1.5 | −1 | 0.5 | −1.87 | 0 |

Here the rounding error is less than 0.01, and thus for all three graphs the function is injective. □

For some other graphs, we see that they have disjoint automorphisms, and therefore they do have quantum symmetries:

4.3.1.3. PROPOSITION. *The graphs $K_{12}$, $C_{12}(5)$, $C_{12}(4,5)$ and $C_{12}(5,6)$ have quantum symmetries.*

PROOF. It is easy to see that the complete graph on $n$ vertices has quantum automorphism group $S_n^+$ and thus, for $n\geq 4$ does have quantum symmetries. For the other graphs, we can check that they have disjoint automorphisms and thus, by Lemma 2.2.2, they have quantum symmetries.

| Graph | Automorphism 1 | Automorphism 2 |
|---|---|---|
| $C_{12}(5)$ | (1 7) | (4 10) |
| $C_{12}(4,5)$ | (1 7)(3 9)(5 11) | (2 8)(4 10)(6 12) |
| $C_{12}(5,6)$ | (1 7) | (4 10) |

□

For $C_{12}(4,5)$, we managed to compute the actual quantum automorphism group. We give the computation in Section 4.5.

4.3.1.4. Proposition. *The graph $C_{12}(2)$ does not have quantum symmetries.*

Proof. We will again show the commutation of $u_{i1}$ with all other generators of $G^+_{aut}(C_{12}(2))$.

First, we consider the vertices in distance 1. These are the vertices $2, 3, 11$ and 12. Since we can write $u_{i1}u_{kl} = u_{i1}u_{kl}\sum_{p\sim 1} u_{ip}$ for a vertex $l \sim 1$, we see that

$$\begin{aligned} u_{i1}u_{k2} &= u_{i1}u_{k2}\left(u_{i1} + u_{i3} + u_{i4} + u_{i12}\right) \\ u_{i1}u_{k3} &= u_{i1}u_{k3}\left(u_{i1} + u_{i2} + u_{i4} + u_{i5}\right) \\ u_{i1}u_{k11} &= u_{i1}u_{k11}\left(u_{i1} + u_{i9} + u_{i10} + u_{i12}\right) \\ u_{i1}u_{k12} &= u_{i1}u_{k12}\left(u_{i1} + u_{i2} + u_{i10} + u_{i11}\right). \end{aligned}$$

Applying Lemma 2.2.5 to these vertices, we see that on the right side, only the term with $u_{i1}$ multiplied from the right does not vanish, and we thus get commutation of $u_{i1}$ and $u_{kl}$ for $l \sim 1$ by Lemma 2.2.1.

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 12 |
| | | 4 | 12 |
| | | 12 | 10 |
| | 3 | 2 | 11 |
| | | 4 | 12 |
| | | 5 | 12 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 11 | 9 | 12 |
| | | 10 | 9 |
| | | 12 | 10 |
| | 12 | 2 | 11 |
| | | 10 | 9 |
| | | 11 | 11 |

Next, we consider vertices in distance 2, which are the vertices $4, 5, 9$ and 10. For these, we can apply Lemma 2.2.4, since we already know that for all vertices $j \sim l$, that $u_{ij}u_{kl} = u_{kl}u_{ij}$:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 4 | 3 | $\{1\}$ |
| | 5 | 6 | $\{1\}$ |
| | 9 | 8 | $\{1\}$ |
| | 10 | 11 | $\{1\}$ |

In distance 3, there are only three vertices, $6, 7$ and 8. Applying Lemma 2.2.4 to them leads to the following table:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 6 | 5 | $\{1\}$ |
| | 7 | 5 | $\{1, 2\}$ |
| | 8 | 6 | $\{1\}$ |

We thus have to consider $u_{i1}u_{k7}$ individually. By the above table, we know that

$$u_{i1}u_{k7} = u_{i1}u_{k7}\left(u_{i1} + u_{i2}\right).$$

Applying Lemma 2.2.6 with $q := 4$ yields that

$$u_{i1}u_{k7}u_{i2} = 0$$

and we thus get

$$u_{i1}u_{k7} = u_{i1}u_{k7}u_{i1} = u_{k7}u_{i1}.$$

Lemma 2.2.6 is applicable since $d(7, 4) = 2$ and we have shown above that for any vertices in distance 2, the generators commute. All in all, we get that all generators commute, and thus $G^+_{aut}(C_{12}(2)) = G_{aut}(C_{12}(2))$. □

4.3.1.5. PROPOSITION. *The graph $C_{12}(4)$ does not have quantum symmetries.*

PROOF. We first note, that for each vertex there is only a single vertex in distance 3, and thus for vertices $i, j, k, l$ with $d(i, k) = 3 = d(j, l)$ we have

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij} = u_{kl}u_{ij}.$$

Next, we will see that $u_{i1}$ commutes with all $u_{kl}$ where $d(l,1) = 1$ by applying Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 1 |
| | | 6 | 9 |
| | | 10 | 5 |
| | 5 | 4 | 8 |
| | | 6 | 9 |
| | | 9 | 1 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 9 | 5 | 1 |
| | | 8 | 4 |
| | | 10 | 5 |
| | 12 | 4 | 8 |
| | | 8 | 4 |
| | | 11 | 1 |

For $d(l,1) = 2$, we can apply Lemma 2.2.4 and get the following:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 3 | 7 | $\{1\}$ |
| | 4 | 3 | $\{1,6\}$ |
| | 6 | 7 | $\{1\}$ |
| | 8 | 7 | $\{1\}$ |
| | 10 | 2 | $\{1,3\}$ |
| | 11 | 7 | $\{1\}$ |

We thus already get commutation for all vertices except for $l = 4$ and $l = 10$. Applying Lemma 2.2.6 with $q := 10$ to 1 and 4 yields $u_{i1}u_{k4}u_{i6} = 0$ and thus we have commutation of $u_{i1}$ and $u_{k4}$. Similarly, Lemma 2.2.6 with $q := 4$ applied to 1 and 10 yields commutation of $u_{i1}$ and $u_{k10}$. All in all, we get that all generators commute and thus $G^+_{aut}(C_{12}(4)) = G_{Aut}(C_{12}(4))$. □

4.3.1.6. PROPOSITION. *The graph $C_{12}(2,6)$ does not have quantum symmetries.*

PROOF. As was the case with $C_{12}(4)$, we can show commutation for all $l$ with $d(l,1) = 1$ by applying Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 1 |
| | | 4 | 5 |
| | | 8 | 3 |
| | | 12 | 3 |
| | 3 | 2 | 4 |
| | | 4 | 5 |
| | | 5 | 2 |
| | | 9 | 2 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 7 | 5 | 6 |
| | | 6 | 2 |
| | | 8 | 3 |
| | | 9 | 8 |
| | 11 | 5 | 12 |
| | | 9 | 10 |
| | | 10 | 2 |
| | | 12 | 3 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 12 | 2 | 4 |
| | | 6 | 2 |
| | | 10 | 2 |
| | | 11 | 1 |

Using Lemma 2.2.5 is also sufficient for vertices $4, 6, 8$ and $10$:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 4 | 7 | 2 |
| | | 8 | 10 |
| | | 9 | 2 |
| | | 11 | 3 |
| | | 12 | 7 |
| | 6 | 2 | 11 |
| | | 3 | 5 |
| | | 9 | 8 |
| | | 10 | 3 |
| | | 11 | 2 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 8 | 3 | 4 |
| | | 4 | 6 |
| | | 5 | 2 |
| | | 11 | 5 |
| | | 12 | 3 |
| | 10 | 2 | 7 |
| | | 3 | 7 |
| | | 5 | 4 |
| | | 6 | 3 |
| | | 7 | 2 |

The only remaining vertices are then 5 and 9. We thus have to consider

$$u_{i1}u_{k5} \quad \text{and} \quad u_{i1}u_{k9}.$$

Multiplying both of these with $1 = \sum_{r \in V(C_{12}(2,6))}$, we get

$$u_{i1}u_{k5} = u_{i1}u_{k5}\left(u_{i1} + u_{i2} + u_{i8} + u_{i9} + u_{i10} + u_{i12}\right)$$
$$u_{i1}u_{k9} = u_{i1}u_{k9}\left(u_{i1} + u_{i2} + u_{i4} + u_{i5} + u_{i6} + u_{i12}\right).$$

Applying Lemma 2.2.5 simplifies this:

| Lemma 2.2.5 | | | | Lemma 2.2.5 | | | |
|---|---|---|---|---|---|---|---|
| $j$ | $l$ | $p$ | $q$ | $j$ | $l$ | $p$ | $q$ |
| 1 | 5 | 2 | 8 | 1 | 9 | 2 | 4 |
| | | 8 | 10 | | | 4 | 6 |
| | | 10 | 3 | | | 6 | 3 |
| | | 12 | 10 | | | 12 | 6 |

to

$$u_{i1}u_{k5} = u_{i1}u_{k5}\left(u_{i1} + u_{i9}\right)$$
$$u_{i1}u_{k9} = u_{i1}u_{k9}\left(u_{i1} + u_{i5}\right).$$

We would now like to apply Lemma 2.2.6 for the last step. However so far, we only know that for any vertex $q$ with $d(q,5) = 1$ the generators $u_{k5}$ and $u_{rq}$ commute. But for all of these vertices, we have $d(q,1) = d(q,9)$, and thus the lemma is not applicable. A similar situation holds for any $q$ with $d(q,9) = 1$.

Consider the following two automorphisms of $C_{12}(2,6)$:

$$\varphi_1 = (1,5)\,(2,4)\,(6,12)\,(7,11)\,(8,10)$$
$$\varphi_2 = (1,9)\,(2,8)\,(3,7)\,(4,6)\,(10,12).$$

It holds that $\varphi_1(1) = 5$ and $\varphi_1(4) = 2$. Since we already know by the above that $u_{i1}u_{k4} = u_{k4}u_{i1}$ for any $i,k \in V(C_{12}(2,6))$, we get by Lemma 2.2.7 that also $u_{i5}u_{k2}$ commute for any $i,k$. We thus can apply Lemma 2.2.6 with $q := 2$ to $1,5$ and $9$ and get

$$u_{i1}u_{k5}u_{i9} = 0,$$

and therefore $u_{i1}$ and $u_{k5}$ commute.

Similarly, we observe that $\varphi_2(1) = 9$ and $\varphi_2(6) = 4$. Again, we already know that $u_{i1}$ and $u_{k6}$ commute for any $i,k$ and thus by Lemma 2.2.7 so do $u_{i9}$ and $u_{k4}$. Then applying Lemma 2.2.6 with $q := 4$ to $1,9$ and $5$, we get that $u_{i1}$ and $u_{k9}$ commute.

We thus showed that $u_{i1}$ commutes with every other generator, and since $C_{12}(2,6)$ is vertex-transitive this is enough to show that $G^+_{aut}(C_{12}(2,6))$ is commutative. □

4.3.1.7. PROPOSITION. *The graph $C_{12}(3,6)$ does not have quantum symmetries.*

PROOF. We first note, that the following permutation is an automorphism of $C_{12}(3,6)$:

$$\varphi := (2,12)\ (3,11)\ (4,10)\ (5,9)\ (6,8).$$

This is the mirroring on the edge that goes through 1 and 7. In particular, we have that $\varphi(1) = 1$, and thus it suffices to show that $u_{i1}$ and $u_{kl}$ commute for $l \in \{2,3,4,5,6,7\}$, since the commutation with the rest of the generators then follows from Lemma 2.2.7 with $\varphi$.

We will now show, that $u_{i1}$ commutes with $u_{k5}$. To see this, we apply Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 5 | 3 | 6 |
| | | 7 | 12 |
| | | 9 | 4 |
| | | 10 | 9 |
| | | 12 | 4 |

It then immediately follows that $u_{i1}$ also commutes with $u_{k9}$ by Lemma 2.2.7 with $\varphi$.

Next, we consider all vertices $l$ such that $d(l,1) = 1$, i.e. $l \in \{2,4,7\}$. We again apply Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 5 | 7 |
| | | 8 | 4 |
| | | 11 | 5 |
| | 4 | 3 | 7 |
| | | 5 | 7 |
| | 7 | 6 | 3 |
| | | 8 | 4 |

We get that $u_{i1}u_{k2}u_{i3} = 0$ by applying Lemma 2.2.6 with $q := 6$. It is applicable since applying Lemma 2.2.7 with the automorphism $v \mapsto v + 1 \mod 12$ yields that $u_{k2}$ and $u_{r6}$ commute, since we already have commutation of $u_{k1}$ and $u_{r5}$. We thus have commutation of $u_{i1}$ and $u_{k2}$.

For $l = 4$, we still need to show that $u_{i1}u_{k4}u_{i7} = 0$ and $u_{i1}u_{k4}u_{i10} = 0$. We have commutation of $u_{k4}$ with both $u_{r8}$ and $u_{r12}$ by Lemma 2.2.7 with the automorphism $v \mapsto v + 3 \mod 12$, since we have commutation of $u_{k1}$ with both $u_{r5}$ and $u_{r9}$. We can thus apply Lemma 2.2.6 to both monomials, once with $q := 8$ and once with $q := 12$, and get that $u_{i1}$ and $u_{k5}$ commute.

For $l = 7$, we need to show that $u_{i1}u_{k7}u_{i4} = 0$ and $u_{i1}u_{k7}u_{i10} = 0$. Again, similar to above, we have commutation of $u_{k7}$ with both $u_{r3}$ and $u_{r11}$ by Lemma 2.2.7 with the automorphism $v \mapsto v + 6$. We can then again apply Lemma 2.2.6 once with $q := 3$ and once with $q := 11$ and get that $u_{i1}$ and $u_{k7}$ commute.

We now have commutation of all generators $u_{i1}$ and $u_{kl}$ where $d(l, 1) = 1$ and thus we also have commutation of all generators $u_{ij}$ and $u_{kl}$ where $d(i, k) = d(j, l) = 1$.

Only missing are now the vertices 3 and 6. For these, applying first Lemma 2.2.4 and then Lemma 2.2.5 shows the commutation of $u_{i1}$ with $u_{k3}$ and $u_{k6}$ :

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 3 | 12 | $\{1, 11\}$ |
| | 6 | 5 | $\{1, 10\}$ |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 3 | 11 | 4 |
| | 6 | 10 | 2 |

We therefore have that all generators commute and we get $G^+_{aut}(C_{12}(3,6)) = G_{aut}(C_{12}(3,6))$.

□

4.3.1.8. PROPOSITION. *The graph $C_{12}(4,6)$ does not have quantum symmetries.*

PROOF. Again we note that we have the following automorphism of $C_{12}(4,6)$:

$$\varphi := (2,12)\ (3,11)\ (4,10)\ (5,9)\ (6,8),$$

i.e. the mirroring on the edge that goes through 1 and 7. As was the case for $C_{12}(3,6)$, it suffices to show that $u_{i1}$ and $u_{kl}$ commute for $l \in \{2,3,4,5,6,7\}$, since the commutation with the rest of the generators then follows from Lemma 2.2.7 with $\varphi$.

We now first consider $u_{i1}$ and $u_{k5}$. Applying Lemma 2.2.5 already yields their commutation:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 5 | 4 | 8 |
| | | 6 | 9 |
| | | 9 | 1 |
| | | 11 | 9 |

Using this, we can show commutation for those vertices $l$ that satisfy $d(1,l) = 2$, i.e. $l \in \{3,4,6\}$. We again apply Lemma 2.2.5 to these vertices to reduce the number of terms we have to consider:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 3 | 10 | 11 |
| | | 12 | 6 |
| | 4 | 2 | 7 |
| | | 7 | 2 |
| | | 9 | 7 |
| | | 11 | 2 |
| | 6 | 3 | 4 |
| | | 4 | 2 |
| | | 9 | 2 |

Starting with vertex 3, we see that

$$u_{i1}u_{k3} = u_{i1}u_{k3}\left(u_{i1} + u_{i5} + u_{i6} + u_{i8}\right).$$

We note, that the automorphism that rotates 1 onto 3 maps 5 to 7 and we thus get $u_{i3}u_{k7} = u_{k7}u_{i3}$ for any $i, k \in V(C_{12}(4, 6))$ by Lemma 2.2.7. We can thus apply Lemma 2.2.6 with $q := 7$ to 1, 3 and 5, since $7 \sim 1$ but $5 \not\sim 7$ and get $u_{i1}u_{k3}u_{i5} = 0$.

Recall, that by Definition 2.2.12, we denote by $\mathrm{CN}(i, j)$ the set of common neighbours of the vertices $i$ and $j$. We now get by Lemma 2.2.13 that $u_{i1}u_{k3}u_{i6} = 0$ and that $u_{i1}u_{k3}u_{i8} = 0$, since $|\mathrm{CN}(1, 3)| = 3$, $|\mathrm{CN}(3, 6)| = 2$ and $|\mathrm{CN}(3, 8)| = 4$. We thus have commutation of $u_{i1}$ and $u_{k3}$.

Looking at vertex 4, we see that we have

$$u_{i1}u_{k4} = u_{i1}u_{k4}\left(u_{i1} + u_{i6}\right).$$

We can apply Lemma 2.2.13 to see that $u_{i1}u_{k4}u_{i6} = 0$, as $|\mathrm{CN}(1, 4)| = 2$ and $|\mathrm{CN}(4, 6)| = 6$. By this, we also get commutation of $u_{i1}$ and $u_{k4}$.

Lastly, looking at vertex 6, we see

$$u_{i1}u_{k6} = u_{i1}u_{k6}\left(u_{i1} + u_{i8}u_{i9} + u_{i11}\right).$$

We will now show, that $u_{k6}$ commutes with $u_{i8}$ and $u_{i9}$. If this is the case, then we get

$$u_{i1}u_{k6}u_{i8} = u_{i1}u_{i8}u_{k6} = 0$$

$$u_{i1}u_{k6}u_{i9} = u_{i1}u_{i9}u_{k6} = 0.$$

Consider the automorphism

$$\sigma = (1\ 6)(2\ 5)(3\ 4)(7\ 12)(8\ 11)(9\ 10).$$

Since $\sigma$ maps 1 to 6, we can translate the question of whether $u_{k6}$ commutes with other generators to the question whether $u_{v1}$ commutes with other generators, for any vertex $v$ by using Lemma 2.2.7. If we thus want to know, whether $u_{k6}$ and $u_{i8}$ commute, we can instead ask, whether $u_{v1}$ and $u_{w11}$ commute. Since we already know that $u_{v1}$ and $u_{w3}$ commute for any vertices $v, w$, and since the mirroring automorphism $\varphi$ from above maps 1 to 1 and 3 to 11, we get that $u_{v1}$ and $u_{w11}$ commute for any $v, w$, again by Lemma 2.2.7. Therefore, also $u_{k6}$ and $u_{i8}$ commute.

Next, to see that $u_{k6}$ and $u_{i9}$ commute, we need to show that $u_{v1}$ and $u_{w10}$ commute. But we already know that $u_{v1}$ and $u_{w4}$ commute and since $\varphi(4) = 10$, we get commutation of $u_{v1}$ and $u_{w10}$ and thus also of $u_{k6}$ and $u_{i9}$.

Lastly, we can apply Lemma 2.2.6 with $q := 2$ to see that

$$u_{i1}u_{k6}u_{i11} = 0.$$

To apply this lemma, we need that $u_{k6}$ commutes with $u_{r2}$ for any vertex $r$. This can easily be seen by using Lemma 2.2.7 with the automorphism $\tau'(v) := v + 1 \mod 12$, as this maps 1 to 2 and 5 to 6.

We thus have that $u_{i1}$ and $u_{k6}$ commutes and thus have commutation of all $u_{ij}$ and $u_{kl}$ where $d(i, k) = d(j, l) = 2$.

We still need to show commutation of $u_{i1}$ with $u_{k2}$ and with $u_{k7}$. Applying Lemma 2.2.4 with $q := 4$ to 1 and 2 yields

$$u_{i1}u_{k2} = u_{i1}u_{k2}\left(u_{i1} + u_{i6}\right).$$

Since we have already seen above that $u_{k2}$ and $u_{i6}$ commute we get

$$u_{i1}u_{k2}u_{i6} = u_{i1}u_{i6}u_{k2} = 0$$

and thus get commutation of $u_{i1}$ and $u_{k2}$.

Applying Lemma 2.2.4 with $q := 4$ to 1 and 7 yields

$$u_{i1}u_{k7} = u_{i1}u_{k7}\left(u_{i1} + u_{i6} + u_{i11}\right).$$

Since there is an automorphism that maps 1 to 6 and 2 to 7, namely the one given by $v \mapsto v + 5 \mod 12$, we can use Lemma 2.2.7 to get that $u_{k7}$ and $u_{i6}$ commute and thus

$$u_{i1}u_{k7}u_{i6} = 0.$$

Similarly, the automorphism $v \mapsto v - 2 \mod 12$ maps 1 to 11 and 7 to 5, and we thus get with Lemma 2.2.7 that $u_{k7}$ and $u_{i11}$ commute and thus

$$u_{i1}u_{k7}u_{i11} = 0.$$

All in all we get that $u_{i1}$ and $u_{k7}$ commute and with that, all of the generators commute and we get $G^+_{aut}(C_{12}(4,6)) = G_{aut}(C_{12}(4,6))$. □

Summarizing the last few propositions, we have the following.

4.3.1.9. Proposition. *The graphs* $C_{12}(2)$*,* $C_{12}(4)$*,* $C_{12}(2,6)$*,* $C_{12}(3,6)$ *and* $C_{12}(4,6)$ *do not have quantum symmetries*

All in all, we thus see that of the twelve circulant graphs on 12 vertices, four have quantum symmetries and eight do not, as can be seen in the following table.

| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $C_{12}$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(3)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(6)$ | $D_{12}$ | $D_{12}$ |
| $K_{12}$ | $S_{12}$ | $S_{12}^+$ |
| $C_{12}(5)$ | $\mathcal{A}$ | ? |
| $C_{12}(4,5)$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $C_{12}(5,6)$ | $\mathcal{A}$ | ? |
| $C_{12}(2)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(4)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(2,6)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(3,6)$ | $D_{12}$ | $D_{12}$ |
| $C_{12}(4,6)$ | $D_{12}$ | $D_{12}$ |

Here we call

$$\mathcal{A} := (\mathbb{Z}_2 \times ((((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2.$$

**4.3.2. Semi-circulant graphs.** As was the case with the circulant graphs, we see that some semi-circulant graphs have disjoint automorphisms and therefore quantum symmetries.

4.3.2.1. PROPOSITION. *The graphs $C_{12}(5^+)$, $C_{12}(3^+,6)$ and $C_{12}(5^+,6)$ have quantum symmetries.*

PROOF. The graphs have the following disjoint automorphisms and thus have quantum symmetries by Lemma 2.2.2.

| Graph | Automorphism 1 | Automorphism 2 |
|---|---|---|
| $C_{12}(5^+)$ | (1 7)(2 8) | (3 9)(4 10) |
| $C_{12}(3^+,6)$ | (2 7)(3 10)(6 11) | (1 8)(4 9)(5 12) |
| $C_{12}(5^+,6)$ | (1 7)(2 8) | (3 9)(4 10) |

□

For $C_{12}(3^+, 6)$, we managed to compute the actual quantum automorphism group. Again, we give the proof in Section 4.5.

4.3.2.2. PROPOSITION. *The graph $C_{12}(2, 5^+)$ does not have quantum symmetries.*

PROOF. We begin by showing that $u_{i1}$ commutes with those generators $u_{kl}$ where $d(l, 1) = 1$, i.e. $l \in \{2, 3, 6, 11, 12\}$. As a first step, we apply Lemma 2.2.5 to these vertices:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 1 |
| | | 9 | 3 |
| | | 12 | 3 |
| | 3 | 2 | 4 |
| | | 5 | 2 |
| | | 8 | 2 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 6 | 5 | 2 |
| | | 7 | 2 |
| | | 8 | 2 |
| | 11 | 9 | 3 |
| | | 10 | 2 |
| | | 12 | 3 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 12 | 2 | 4 |
| | | 7 | 2 |
| | | 10 | 2 |
| | | 11 | 1 |

From the above, we already see that $u_{i1}$ and $u_{k12}$ commute. By Lemma 2.2.7 we get commutation of all $u_{ij'}$ and $u_{kl'}$ where $j'$ is uneven and $l' = j' - 1 \mod 12$, using the automorphism $v \mapsto v + 2 \mod 12$ repeatedly for the lemma.

Next, we note that for the commutation of $u_{i1}$ and $u_{k2}$, the only thing missing is that

$$u_{i1}u_{k2}u_{i4} = 0.$$

This can be seen using Lemma 2.2.6 with $q := 3$ and the fact that $u_{k2}$ and $u_{r3}$ commute by the above.

Similarly, we only need

$$u_{i1}u_{k3}u_{i4} = 0$$

to see that $u_{i1}$ and $u_{k3}$ commute and can see that again with Lemma 2.2.6 with $q := 2$.

In order to see that $u_{i1}$ and $u_{k6}$ commute, we need that

$$u_{i1}u_{k6}u_{i4} = 0.$$

By Lemma 2.2.7 with the automorphism

$$\varphi = (1\ 6)(2\ 5)(3\ 4)(7\ 12)(8\ 11)(9\ 10)$$

we get that $u_{k6}$ and $u_{i4}$ commute, since $\varphi(1) = 6$ and $\varphi(3) = 4$ and we have already shown commutation of $u_{k1}$ and $u_{i3}$. We thus get

$$u_{i1}u_{k6}u_{i4} = u_{i1}u_{i4}u_{k6} = 0$$

as desired.

For the commutation of $u_{i1}$ and $u_{k11}$ we need

$$u_{i1}u_{k11}u_{i4} = 0.$$

Since we already know that $u_{k11}$ and $u_{r10}$ commute, we can apply Lemma 2.2.6 with $q := 10$ to achieve this. We thus have commutation of all generators $u_{ij}$ and $u_{kl}$ where $d(j, l) = 1$.

For some vertices $l$ with $d(l, 1) = 2$ it is enough to apply Lemma 2.2.4 first and then Lemma 2.2.5 to get that the missing monomial is 0:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 4 | 3 | $\{1, 8\}$ |
| | 5 | 6 | $\{1, 8\}$ |
| | 10 | 5 | $\{1, 2\}$ |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 4 | 8 | 2 |
| | 5 | 8 | 2 |
| | 10 | 2 | 6 |

For the other three vertices, we also first apply Lemma 2.2.4:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 7 | 6 | $\{1, 4\}$ |
| | 8 | 6 | $\{1, 4, 5\}$ |
| | 9 | 7 | $\{1, 3, 4\}$ |

To get that $u_{i1}u_{k7}u_{i4} = 0$ we apply Lemma 2.2.6 with $q := 5$.

Next, we consider the automorphism

$$\sigma = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7).$$

We see that $\sigma(1) = 12$ and $\sigma(5) = 8$ and thus with Lemma 2.2.7 we get that $u_{k8}$ and $u_{r12}$ commute. To see now that both $u_{i1}u_{k8}u_{i4} = 0$ and $u_{i1}u_{k8}u_{i5} = 0$ hold, we can apply Lemma 2.2.6 with $q := 12$ to each of those monomials. With the same automorphism $\sigma$, Lemma 2.2.7 shows that $u_{k9}$ and $u_{r12}$ commute, and again, we see that $u_{i1}u_{k9}u_{i3} = 0$ and $u_{i1}u_{k9}u_{i4} = 0$ hold by applying Lemma 2.2.6 with $q := 12$ to both monomials.

All in all we see that all generators commute with $u_{i1}$ and thus with Corollary 2.2.8 we get $G^+_{aut}(C_{12}(2, 5^+)) = G_{aut}(C_{12}(2, 5^+))$. □

4.3.2.3. PROPOSITION. *The graph $C_{12}(4, 5^+)$ does not have quantum symmetries.*

PROOF. For the vertices $l \in \{3, 10, 11, 12\}$, it is sufficient to apply Lemma 2.2.5 to show that $u_{i1}$ and $u_{kl}$ commute:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 3 | 5 | 2 |
| | | 6 | 12 |
| | | 9 | 12 |
| | | 10 | 12 |
| | | 12 | 4 |
| | 10 | 3 | 1 |
| | | 4 | 2 |
| | | 7 | 2 |
| | | 8 | 2 |
| | | 12 | 4 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 11 | 2 | 10 |
| | | 5 | 10 |
| | | 6 | 10 |
| | | 8 | 2 |
| | | 9 | 8 |
| | 12 | 4 | 6 |
| | | 7 | 2 |
| | | 8 | 2 |
| | | 11 | 1 |

We will now take a look at the rest of the vertices in distance 2 to the vertex 1, i.e. $l \in \{4, 7, 8\}$. To simplify, we will again first apply Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 4 | 2 | 5 |
| | | 6 | 7 |
| | | 7 | 2 |
| | 7 | 4 | 2 |
| | | 5 | 4 |
| | | 9 | 6 |
| | 8 | 6 | 9 |
| | | 10 | 11 |
| | | 11 | 1 |

Next, we want to show that $u_{i1}u_{k4}u_{i9} = 0$ and $u_{i1}u_{k4}u_{i10} = 0$ hold. For this, we first observe that the permutation

$$\varphi = (1\ 4)(2\ 3)(5\ 12)(6\ 11)(7\ 10)(8\ 9)$$

is an automorphism of $C_{12}(4, 5^+)$. By Lemma 2.2.7 with $\varphi$ we thus get that $u_{k4}$ and $u_{r2}$ commute. We can thus apply Lemma 2.2.6 to both monomials with $q := 2$ to get the desired result and see that $u_{i1}$ and $u_{k4}$ commute.

For $l = 7$, we want to show that $u_{i1}u_{k7}u_{i2} = 0 = u_{i1}u_{k7}u_{i10}$. Similarly to above, we note that

$$\sigma = (1\ 7)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)$$

is an automorphism of $C_{12}(4, 5^+)$ and thus by Lemma 2.2.7 we get commutation of $u_{k7}$ and $u_{r9}$. Then, by Lemma 2.2.6 with $q := 9$, we get the desired result and thus $u_{i1}$ and $u_{k7}$ commute.

For $l = 8$, we need to show that $u_{i1}u_{k8}u_{i2} = 0 = u_{i1}u_{k8}u_{i5}$. In this case, we use the automorphism of $C_{12}(4, 5^+)$

$$\tau = (1\ 8)(2\ 7)(3\ 6)(4\ 5)(9\ 12)(10\ 11).$$

Using this with Lemma 2.2.7, we see that $u_{k8}$ and $u_{i6}$ commute and can thus apply Lemma 2.2.6 with $q := 6$ to get the desired outcome. We thus get that $u_{i1}$ and $u_{k8}$ commute and with this, all generators $u_{ij}$ and $u_{kl}$ commute where $d(j, l) = 2$.

For the remaining vertices $l$ with $d(l, 1) = 1$, we now apply Lemma 2.2.4 and, where necessary, Lemma 2.2.5 to see that the remaining monomial also vanishes:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 5 | 12 | $\{1,4\}$ |
| | 6 | 12 | $\{1,7\}$ |
| | 9 | 12 | $\{1,8\}$ |
| | 2 | 12 | $\{1\}$ |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 5 | 4 | 6 |
| | 6 | 7 | 2 |
| | 9 | 8 | 2 |

We therefore see that all generators commute and we get $G^+_{aut}(C_{12}(4,5^+)) = G_{aut}(C_{12}(4,5^+))$. □

All in all, the following table summarises the results for semi-circulant graphs:

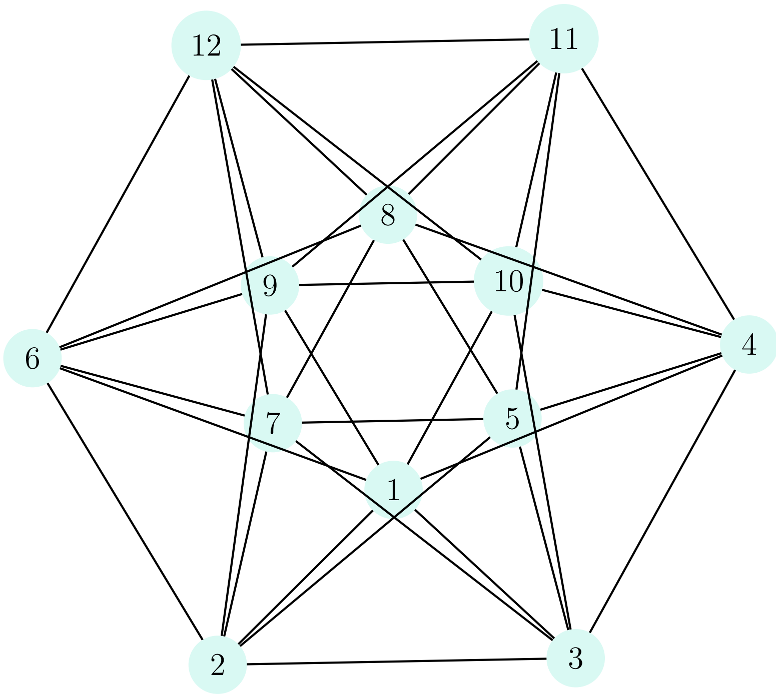
| Graph | Automorphism Group | Quantum Automorphism Group |
|---|---|---|
| $C_{12}(5^+)$ | $\mathbb{Z}_2 \times S_4$ | ? |
| $C_{12}(3^+,6)$ | $H_2 \times S_3$ | $H_2^+ \times S_3$ |
| $C_{12}(5^+,6)$ | $\mathbb{Z}_2 \times S_4$ | ? |
| $C_{12}(2,5^+)$ | $D_6$ | $D_6$ |
| $C_{12}(4,5^+)$ | $D_6$ | $D_6$ |

## 4.4. Special Cases

**4.4.1. The Icosahedron.** It was shown by Schmidt in his PhD thesis that the Icosahedron does not have quantum symmetries.

4.4.1. Proposition ([75]). *The Icosahedron does not have quantum symmetries.*

**4.4.2. The line graph of $C_6(2)$.**




4.4.1. PROPOSITION. *The graph $L(C_6(2))$ does not have quantum symmetries.*

PROOF. The commutation of $u_{i1}$ with $u_{kl}$ for $l \in \{4,5,6,7,8,11,12\}$ can be seen by just applying Lemma 2.2.5:

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 4 | 3 | 9 |
| | | 5 | 7 |
| | | 8 | 2 |
| | | 10 | 2 |
| | | 11 | 5 |
| | 5 | 6 | 3 |
| | | 9 | 12 |
| | | 10 | 6 |
| | | 12 | 1 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 6 | 2 | 10 |
| | | 7 | 5 |
| | | 8 | 2 |
| | | 9 | 3 |
| | | 12 | 7 |
| | 7 | 4 | 2 |
| | | 9 | 4 |
| | | 10 | 11 |
| | | 11 | 1 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 8 | 2 | 10 |
| | | 3 | 9 |
| | | 9 | 3 |
| | | 10 | 2 |
| | 11 | 2 | 7 |
| | | 3 | 6 |
| | | 6 | 3 |
| | | 7 | 1 |

Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 12 | 2 | 4 |
| | | 3 | 5 |
| | | 4 | 2 |
| | | 5 | 1 |

The only vertices $l$ for which we still need to show commutativity of $u_{i1}$ and $u_{kl}$ are thus $l \in \{2,3,9,10\}$. However, since we have the following three automorphisms in $G_{aut}(L(C_6(2)))$, we only need to show it for one such $l$ and the rest follows immediately by Lemma 2.2.7:

$$\varphi = (2\ 3)(4\ 6)(5\ 7)(9\ 10)(11\ 12)$$
$$\sigma = (2\ 9)(3\ 10)(5\ 11)(7\ 12)$$
$$\tau = (2\ 10)(3\ 9)(4\ 6)(5\ 12)(7\ 11).$$

We will therefore now show commutation of $u_{i1}$ and $u_{k2}$. We do this, by first applying Lemma 2.2.4 and then using Lemma 2.2.5 to see that the remaining monomials vanish:

Lemma 2.2.4

| $j$ | $l$ | $q$ | $p$ |
|---|---|---|---|
| 1 | 2 | 4 | $\{1, 3, 5\}$ |

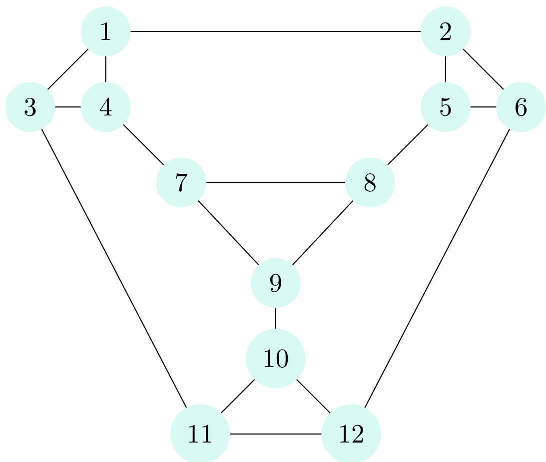
Lemma 2.2.5

| $j$ | $l$ | $p$ | $q$ |
|---|---|---|---|
| 1 | 2 | 3 | 6 |
| | | 5 | 6 |

All in all we thus get that $G^+_{aut}(L(C_6(2))) = G_{aut}(L(C_6(2)))$. □

**4.4.3. The truncated tetrahedron.** We want to show, that $C(G^+_{aut}(Trunc(K_4)))$ is commutative, i. e. that $G_{aut}(Trunc(K_4)) = G^+_{aut}(Trunc(K_4))$.

4.4.1. Theorem. *The truncated tetrahedron, $\Gamma = Trunc(K_4)$, shown below, does not have quantum symmetry, i. e. $C(G^+_{aut}(\Gamma))$ is commutative.*




Before we prove the above theorem, we want to collect some properties of $\Gamma$.

4.4.2. Properties.

- $\Gamma$ *has diameter* 3, *i.e. the largest distance between two vertices is* 3.
- $\Gamma$ *is* 3*-regular, i.e. every vertex has exactly* 3 *neighbours.*
- $\Gamma$ *consists of* 4 *triangles.*
- *Each vertex of* $\Gamma$ *is connected exactly to two vertices in the same triangle and one vertex in a different triangle.*

We now want to introduce some notation to make talking about the different triangles easier.

4.4.3. DEFINITION.

(i) We denote by $\mathcal{T}$ the set of the four triangles that make up $\Gamma$.
(ii) Let $v \in V(\Gamma)$. We denote by $T(v)$ the triangle containing $v$.
(iii) Let $v \in V(\Gamma)$. We denote by $AT(v)$ the unique triangle that is *adjacent* to $v$, i.e. that is connected to $v$ by an edge but is different from $T(v)$.
(iv) Let $v \in V(\Gamma)$. We denote by $\mathcal{NT}(v) := \mathcal{T} \setminus \{T(v), AT(v)\}$ the set of *non-adjacent triangles* of $v$.

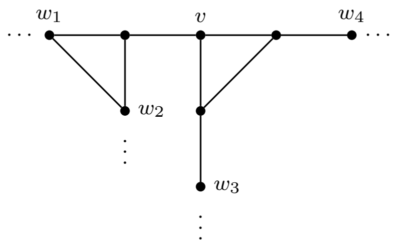
4.4.4. LEMMA. *Let $v, w \in V(\Gamma)$ be two vertices. If they are in the same triangle, then their adjacent triangles are distinct:*

$$T(v) = T(w) \Longrightarrow AT(v) \neq AT(w).$$

4.4.5. LEMMA. *Let $v, w \in V(\Gamma)$ be two vertices with $d(v, w) = 2$. Then it holds that either $v$ is in the adjacent triangle of $w$ or vice versa:*

$$d(v, w) = 2 \Longrightarrow v \in AT(w) \vee w \in AT(v).$$

PROOF. If $v$ is not in $AT(w)$, then in order to reach $v$ from $w$, one step has to be made inside $T(w)$. Since $d(v, w) = 2$, the next step, which is the first step out of $T(w)$, will have to reach $v$ already. Thus, by the definition of $AT(v)$, the triangle we just left is $AT(v)$. Since it is also $T(w)$ we get $w \in AT(v)$. The proof can be visualized with the following sketch:




□

4.4.6. LEMMA. *Let $v \in V(\Gamma)$ be a vertex. All vertices $w \in V(\Gamma)$ that fulfill $d(v, w) = 3$ are in the triangles in $\mathcal{NT}(v) = \mathcal{T} \setminus \{T(v), AT(v)\}$.*

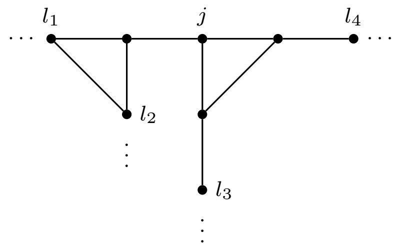
PROOF. Indeed, if $w$ was in $T(v)$, then the distance of $w$ to $v$ would be 1. On the other hand if $w \in AT(v)$ holds, then either $d(v, w) = 1$, if $w$ is the unique vertex in $AT(v)$ that shares an edge with $v$, or $d(v, w) = 2$, as $w$ is connected to the vertex in $AT(v)$ that shares an edge with $v$. □

We now come to the proof of Theorem 4.4.1.

PROOF. Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. We want to show that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $i, j, k, l \in V(\Gamma)$. By Lemma 2.2.3 it suffices to show the statement for $i, j, k, l$ such that $d(i, k) = d(j, l)$.

Step 1: $d(i,k) = d(j,l) = 1$: Since $\Gamma$ does not contain any quadrangles, by Lemma 2.2.9 we have $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ and therefore $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $d(i,k) = d(j,l) = 1$.

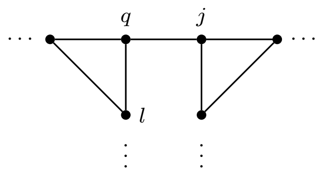
Step 2: $d(i,k) = d(j,l) = 2$. Let us fix $j \in V(\Gamma)$. Then there are four vertices at distance 2 from $j$: two of them are in $AT(j)$, reached by following the edge into $AT(j)$ and then following either edge within the triangle, and one is in each of the triangles in $\mathcal{NT}(j)$, reached by following either edge within $T(j)$ and then taking the single edge that leads out of $T(j)$. Let us sketch the situation:




In this situation, $l_1$ and $l_2$ are the vertices at distance 2 that are in $AT(j)$ and $l_3$ and $l_4$ are each in one of the triangles of $\mathcal{NT}(j)$. Note, that while $l_3$ and $l_4$ are not in $AT(j)$, it holds that $j \in AT(l_3)$ and $j \in AT(l_4)$ by Lemma 4.4.5. Let now $l$ be one of $l_1$ and $l_2$, i.e. $l$ is a vertex such that $d(j,l) = 2$ and $l \in AT(j)$. We will show, that for all $i,k \in V(\Gamma)$ with $d(i,k) = 2$ the following holds:

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}.$$

Then, by Lemma 2.2.1, we have $u_{ij}u_{kl} = u_{kl}u_{ij}$. We want to use Lemma 2.2.4, with distance $m = 2$ and choosing $s = t = 1$, which we can do since $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$, we want to find a vertex $q$ for which it holds that $d(j,q) = 1$ and $d(q,l) = 1$. If we choose $q$ such that the only vertex $p$ fulfilling $d(p,l) = 2$ and $d(p,q) = 1$ is $j$, then we get the desired result. Let us again take a look at the situation we have.

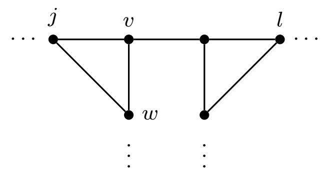



Choosing $q$ as indicated in the sketch above, we see that there are three vertices adjacent to $q$, one of which is $l$ and one is $j$. The third vertex is in the same triangle as $l$ and thus is at distance 1 from $l$. Therefore, the only vertex adjacent to $q$ and at distance 2 from $l$ is in fact $j$. Lemma 2.2.4 yields

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;\, d(p,l)=2 \\ (p,q)\in E(\Gamma)}} u_{ip} = u_{ij}u_{kl}u_{ij}.$$

Thus, for all $i,j,k,l \in V(\Gamma)$ with $d(i,k) = d(j,l) = 2$ and $l \in AT(j)$, $u_{ij}$ and $u_{kl}$ commute. As noted above, it holds that if $l \notin AT(j)$, we have $j \in AT(l)$. Therefore, we get that $u_{ij}$ and $u_{kl}$ commute for all $i,j,k,l$ with $d(i,k) = d(i,j) = 2$.

Step 3: $d(i,k) = d(j,l) = 3$. Fixing again $j \in V(\Gamma)$, we know by Lemma 4.4.6 that all vertices $l$ with $d(j,l) = 3$ are in the triangles in $\mathcal{NT}(j)$, in other words $l \notin \{T(j), AT(j)\}$. Note moreover, that $\mathcal{NT}(j) = \{AT(v), AT(w)\}$ if $T(j) = \{j, v, w\}$. Indeed, there are 4 triangles in total, and since $T(j) = T(v)$ implies $AT(j) \neq AT(v)$, we can write the set of all triangles as $\mathcal{T} = \{T(j), AT(j), AT(v), AT(w)\}$. Let now $l$ be a fixed vertex at distance 3 from $j$ and let $v \in T(j)$ be the vertex that satisfies $l \in AT(v)$. We are thus in the situation sketched below.




In this situation, we can use Lemma 2.2.4 with $s = 1$, $t = 2$, putting $q := v$, to get for all $i, k \in V(\Gamma)$, $d(i,k) = 3$ that

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;\, d(p,l)=3 \\ (p,q) \in E(\Gamma)}} u_{ip}.$$

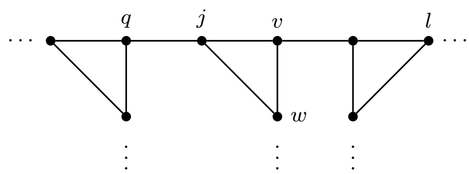
Note that we can apply Lemma 2.2.4 here as we have shown above, that $u_{xy}$ and $u_{x'y'}$ commute with $d(x,x') = d(y,y') = 2$ and thus in particular for all $a$ with $d(a,k) = 2$ it holds that $u_{kl}$ and $u_{aq}$ commute.

Since we have $d(w,l) = 3$ we get that $j$ and $w$ are connected to $q$ and at distance 3 to $l$ and Lemma 2.2.4 thus yields

$$u_{ij}u_{kl} = u_{ij}u_{kl}\left(u_{ij} + u_{iw}\right).$$

It remains to show that we have $u_{ij}u_{kl}u_{iw} = 0$ for all $d(i,k) = 3$, since then $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$ and thus $u_{ij}$ and $u_{kl}$ commute by Lemma 2.2.1. We want to apply Lemma 2.2.5 to $i, j, k, l$ and $w$ in order to show this. We thus have $m = 3$. The lemma is applicable, since we have $w \neq j$ and $d(w,l) = 3$. We want to find a vertex $q$ and a distance $s$, such that $d(q,l) = s$ and $d(j,q) \neq d(q,w)$ and that moreover fulfills that $l$ is the only vertex satisfying $d(q,l) = s$, $d(l,j) = 3$ and $d(l,p) = 3$. If we find such a $q$ and $s$ we are done, since then, by Lemma 2.2.5, $u_{ij}u_{kl}u_{iw} = 0$.

We put $q$ as the unique vertex in $AT(j)$ that is adjacent to $j$ and $s := d(q,l)$. We are thus in the situation sketched below:



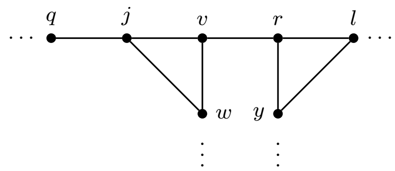

Then obviously $s = d(q,l)$ holds. Moreover, $1 = d(j,q) \neq d(w,q) = 2$ holds, since $T(q) \in \mathcal{NT}(w)$. It remains to show, that for any vertex $x \in V(\Gamma)$ the following holds:

$$d(x,q) = s \text{ and } d(x,j) = 3 \text{ and } d(x,w) = 3 \Longrightarrow x = l.$$

Let therefore such a vertex $x$ be given. We claim that $x \in AT(v)$.

Indeed, if $x \notin AT(v)$, then either $x$ would have to be in $T(v)$ or $T(x)$ would have to be in $\mathcal{NT}(v)$. But if $x \in T(v) = T(j)$, then $d(x,j) \leq 1$ which is a contradiction to $d(x,j) = 3$. If on the other hand we have $T(x) \in \mathcal{NT}(v) = \{AT(j), AT(w)\}$, then we have either $x \in AT(j)$ or $x \in AT(w)$. If $x \in AT(j)$, then $d(x,j) \leq 2$ as was argued above, which is again a contradiction to $d(x,j) = 3$. If $x \in AT(w)$ we have by the same argument that $d(x,w) \leq 2$, which is a contradiction to $d(x,w) = 3$.

We thus have $x \in AT(v) = T(l)$. Let us look at another sketch of the situation, in order to name the relevant vertices.




We know so far, that $x \in \{r, y, l\}$. Since however $d(j,r) = d(w,r) = 2 \neq 3$ we can narrow it down to $x \in \{y, l\}$. It remains to show that $d(q,l) \neq d(q,y)$, since then we have shown that $x = l$. In the sketch above, we see that $T(q) \in \mathcal{NT}(r)$ and thus we have either $q \in AT(l)$ or $q \in AT(y)$. We will show the claim for $q \in AT(l)$, the other situation can be shown similarly. If $q$ is in $AT(l)$, then we have $d(q,l) \leq 2$, and since the unique vertex connected to $q$ that is in a different triangle is already $j$, we even know $d(q,l) = 2$. We will now show, that $d(q,y) = 3$. Since $\Gamma$ has diameter 3, we already know, that $d(q,y) \leq 3$. Since a shortest path from $l$ to $q$ is of length 2, any path from $y$ to $q$ via $l$ will have at least length 3 and there is also a path of length 3 via $l$. Moreover, any path from $y$ to $q$ via $r$ will be at least of length 4, as can be seen in the sketch above. The last possibility to reach $q$ from $y$ is thus by first traversing $AT(y)$. But since $q \notin AT(y)$, which is the case as $q \in AT(l)$, any such path will consist of first taking the edge from $y$ to $AT(y)$, then taking one edge within $AT(y)$ and then leaving $AT(y)$ via a third edge, and thus such a path is at least of length 3 again. We thus conclude $d(q,l) = 2 \neq 3 = d(q,y)$.

Therefore, we can apply Lemma 2.2.5 and get

$$u_{ij} \left( \sum_{\substack{x;\, d(x,j)=d(x,w)=3 \\ d(q,x)=d(q,l)}} u_{kx} \right) u_{iw} = u_{ij} u_{kl} u_{iw} = 0.$$

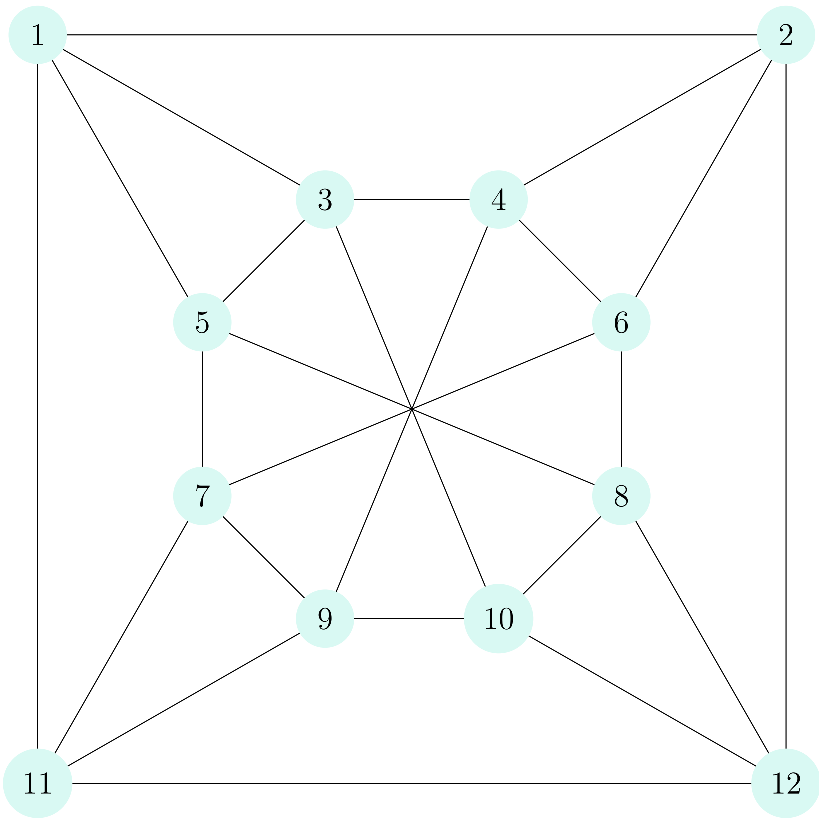
We thus conclude that $u_{ij}$ and $u_{kl}$ commute for any $i,j,k,l \in V(\Gamma)$ such that $d(i,k) = d(j,l) = 3$.

Since the diameter of $\Gamma$ is 3, we now know that all generators of $C(G^+_{aut}(\Gamma))$ commute and thus $G^+_{aut}(\Gamma) = G_{aut}(\Gamma)$.

$\square$

**4.4.4. The distance 3 graph of the truncated tetrahedron.** In this section, we will show that the distance 3 graph of the truncated tetrahedron, $Antip(Trunc(K_4))$, does not have quantum symmetry.

4.4.1. Theorem. *The distance* 3 *graph of the truncated tetrahedron as shown below,* $\Gamma = Antip(Trunc(K_4))$*, does not have quantum symmetry, i.e.* $C(G^+_{aut}(\Gamma))$ *is commutative.*




We collect some properties about the graph before proving the theorem.

4.4.2. Properties.

- $\Gamma$ *has diameter* 2*.*
- $\Gamma$ *is* 4*-regular.*
- *The graph is made up of* 4 *disjoint triangles, i.e. for each vertex* $v \in V$*, it holds that there are two other vertices* $w$ *and* $x$*, such that* $\{v, w, x\}$ *forms a triangle and for all other vertices* $y \in V \setminus \{v, w, x\}$ *it holds that if* $(v, y) \in E$ *then* $v$ *and* $y$ *have no common neighbour. We will call this triangle* $T(v)$*.*
- *Every vertex* $v$ *is also part of three distinct quadrangles. On the one hand, it is part of one containing vertices from all* 4 *triangles and on the other hand it is part of two quadrangles each containing* 2 *vertices from* $T(v)$ *and* 2 *vertices from another triangle. Both of the latter vertices share a triangle however.*
- *All vertices that are not connected have either one or two common neighbours.*

- *For every pair of vertices that are in the same triangle, there is a unique quadrangle containing the same two vertices.*
- *For every pair of vertices that are in distance* 2*, if they are in a quadrangle, then this quadrangle is unique.*
- *For every pair of adjacent vertices that do not share a triangle, there are* 2 *quadrangles containing both vertices.*

4.4.3. DEFINITION.

(i) If $v \in V$ is a vertex, we denote the unique triangle containing $v$ as $T(v)$.

(ii) If $v, w \in V$ and $d(v, w) = 2$, we want to count the number of distinct triangles connected by a possible quadrangle containing $v$ and $w$. For this, we first note, that if a quadrangle $\mathcal{Q}(v, w)$ containing $v$ and $w$ exists, then it is unique, i.e. there are no two distinct quadrangles containing both $v$ and $w$, for $d(v, w) = 2$, as was noted in the Properties 4.4.2 above. We now put for an existing quadrangle $\mathcal{Q}(v, w)$ consisting of the vertices $\{v, w, x, y\}$ the set of triangles connected by this quadrangle as $\mathcal{T}(\mathcal{Q}(v, w)) := \{T(v), T(w), T(x), T(y)\}$. With this, we define

$$Q(v, w) = \begin{cases} 0, & \text{if } v \text{ and } w \text{ do not share a quadrangle} \\ |\mathcal{T}(\mathcal{Q}(v, w))| & \text{otherwise.} \end{cases}$$

Note, that by the properties of $\Gamma$ listed above, we have that $Q(v, w) \in \{0, 2, 4\}$ for all vertices $v$ and $w$ at distance 2.

4.4.4. LEMMA. *Let $v$ and $w$ be two vertices in $\Gamma$ such that $d(v, w) = 2$ and $Q(v, w) = 0$. Then the unique common neighbour $p$ of $v$ and $w$ fulfills that we have either $p \in T(v)$ or $p \in T(w)$.*

PROOF. The fact that $p$ exists comes from the fact, that $d(v, w) = 2$ and therefore, there is a path of length 2 from $v$ to $w$. Since however there is no quadrangle containing both $v$ and $w$, there can not be two common neighbours.

The fact, that $p \in T(v)$ or $p \in T(w)$ holds, comes from the fact that each vertex has 4 neighbours, two of which are in the same triangle as the vertex itself, and the other two are in two distinct different triangles. Thus, if we take one edge incident to $v$ that leads to a triangle different from $T(v)$, the vertex we reach has only one neighbour apart from $v$ that is in a different triangle than itself. Therefore, there is only one option to take another edge to a completely new triangle. Repeating this twice more, we again reach $v$, and since each time there was only one choice, we have traversed the unique quadrangle containing $v$ and connecting all 4 triangles. Since $v$ and $w$ do not share a quadrangle however, none of the vertices we traversed was $w$, and thus on the path from $v$ to $w$, at least one of the two steps has to be made within the same triangle. $\square$

4.4.5. LEMMA. *Let $v \neq w$ be two vertices in $\Gamma$ such that $T(v) = T(w)$, in particular we have $v \sim w$. There is at most one vertex $s \in V$ such that $d(v,s) = 2$, $Q(v,s) = 0$ and $w \sim s$ hold.*

PROOF. Observe, that both $v$ and $w$ have 4 neighbours, as $\Gamma$ is 4-regular, and 2 of these neighbours are in the same triangle. Thus, each of $v$ and $w$ have two neighbours in triangles that are distinct from $T(v) = T(w)$. We note, that as soon as $v$ has a neighbour $t$ in the same triangle as the neighbour $s$ of $w$, we have that $\{v, w, s, t\}$ forms a quadrangle, as $s \sim t$ follows from the fact that they share a triangle. Note moreover, that $s = t$ can not be the case, since otherwise $\{v, w, s\}$ would form a triangle, which contradicts $T(s) \neq T(v)$. Therefore, we have that $d(v,t) = 2$ and $Q(v,t) > 0$.

If we have on the other hand that as soon as a neighbour $s$ of $w$ is in a triangle, in which $v$ does not have a neighbour, then $d(v,s) = 2$ and $Q(v,s) = 0$.

However, as there are only 3 triangles in $\Gamma$ distinct from $T(v)$, and as both $v$ and $w$ have each 2 neighbours in distinct triangles, at least one pair of neighbours has to be in the same triangle. Therefore, at most one neighbour $s$ of $v$ fulfills $d(v,s) = 2$ and $Q(v,s) = 0$. □

PROOF. of Theorem 4.4.1 Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. We will prove that $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $i, j, k, l \in V(\Gamma)$ in several steps. Recall, that by Lemma 2.2.3, we only need to consider the case where $d(i,k) = d(j,l)$, since otherwise the product is already 0 and thus the generators commute.

Step 1: Let first $d(i,k) = d(j,l) = 1$. Let now $i$ and $k$ be in the same triangle and let also $j$ and $l$ share a triangle. Then, by the Properties 4.4.2 of $\Gamma$, $i, j, k$ and $l$ fulfill the premise of Lemma 2.2.11. We thus get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 2.2.11.

If we have that one of the pairs of vertices $\{i, k\}$ or $\{j, l\}$ share a triangle and the other does not, we are in the premise of Corollary 2.2.15 and therefore we get $u_{ij}u_{kl} = u_{kl}u_{ij} = 0$ for these choices of $i, j, k$ and $l$.

For $d(i,k) = d(j,l) = 1$, we now still need to show the case where neither $i$ and $k$ nor $j$ and $l$ share a triangle. However, in order to show the claim for this case, we will first need to consider the case of distance 2.

Next, we will consider vertices, such that $d(i,k) = d(j,l) = 2$. Here we will make a case distinction on $Q(i,k)$ and $Q(j,l)$ as defined above in Definition 4.4.3.

Step 2: We first consider the cases where $Q(i,k) \neq Q(j,l)$. In these cases, we will prove $u_{ij}u_{kl} = 0$. Let therefore $Q(i,k) = 2$, $Q(j,l) = 4$. Let $q$ be a common neighbour of $j$ and $l$. Then $T(q) \notin \{T(j), T(l)\}$ since $Q(j,l) = 4$ and $q$ lies in the quadrangle connecting $j$ and $l$. For any common neighbour $s$ of $i$ and $k$ however, we have either $s \in T(i)$ or $s \in T(k)$, since $s$ is in the quadrangle containing $i$ and $k$ and as $Q(i,k) = 2$ there are vertices from only two distinct triangles in this quadrangle, which are $T(i)$ and $T(k)$. Since $i$ and $k$ have exactly 2 common neighbours, we call them $s_1$ and $s_2$, where $s_1 \in T(i)$ and $s_2 \in T(k)$. We thus get

$$u_{ij}u_{kl} = u_{ij}\sum_{s\in V} u_{sq}u_{kl} = u_{ij}\sum_{\substack{s\in V\\ s\sim i\\ s\sim k}} u_{sq}u_{kl} = u_{ij}u_{s_1q}u_{kl} + u_{ij}u_{s_2q}u_{kl}. \tag{4.4.1}$$

Since however, by Step 1, we know that $u_{ab}u_{cd} = 0$ if $T(a) = T(c)$ but $T(b) \neq T(d)$ or vice versa, we get

$$u_{ij}u_{s_1q} = 0$$

since $q \notin T(j)$ and

$$u_{s_2q}u_{kl} = 0$$

since $q \notin T(l)$. This together with 4.4.1 yields

$$u_{ij}u_{kl} = 0.$$

The same argument can be used to show that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 4$ and $Q(j,l) = 2$.

Step 3: Let now $Q(i,k) = 0$ and $Q(j,l) = 4$. Since $Q(i,k) = 0$, there is no quadrangle containing both $i$ and $k$. Then $i$ and $k$ have exactly one common neighbour, let us call it $p$. By Lemma 4.4.4, we have, that either $p \in T(i)$ or $p \in T(k)$. Let now $q$ be one of the common neighbours of $j$ and $l$. As was argued in Step 2, we have that $T(q) \notin \{T(j), T(l)\}$. We get

$$u_{ij}u_{kl} = u_{ij}\sum_{s\in V} u_{sq}u_{kl} = u_{ij}u_{pq}u_{kl}$$

as $q$ is a common neighbour of $j$ and $l$ and $p$ is the only common neighbour of $i$ and $k$. If now $q \in T(i)$, we get by Step 1 that

$$u_{ij}u_{pq} = 0$$

as $p \notin T(j)$ and similarly, if $q \in T(k)$ we get

$$u_{pq}u_{kl} = 0.$$

Thus, we get

$$u_{ij}u_{kl} = u_{ij}u_{pq}u_{kl} = 0.$$

We can show similarly, that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 4$ and $Q(j,l) = 0$.

Step 4: Let $Q(i,k) = 0$ and $Q(j,l) = 2$ and let $p$ be the unique common neighbour of $i$ and $k$. We argue as in Step 3 that either $p \in T(i)$ or $p \in T(k)$ holds. We know moreover that $j$ and $l$ have two common neighbours, let us call them $q_1$ and $q_2$. We assume without loss of generality, that $q_1 \in T(j)$ and $q_2 \in T(l)$. We know by Step 1, that $p \in T(i)$ implies $u_{ij}u_{pq_1} = u_{pq_1}u_{ij}$, while $p \in T(k)$ implies $u_{kl}u_{pq_2} = u_{pq_2}u_{kl}$. In both cases, we can apply Lemma 2.2.13 to get that $u_{ij}u_{kl} = 0$. Similarly, we get that $u_{ij}u_{kl} = 0$ if $Q(i,k) = 2$ and $Q(j,l) = 0$ holds.

All in all, we see that

$$u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$$

holds whenever we have $Q(i,k) \neq Q(j,l)$.

Step 5: We now consider the three cases for $Q(i,k) = Q(j,l)$. Let first $Q(i,k) = Q(j,l) = 4$. Let now $p \in V \setminus \{j\}$ be a vertex that is distinct from $j$. We want to show that for any such $p$, we have

$$u_{ij}u_{kl}u_{ip} = 0$$

since then we get

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{v\in V} u_{iv} = u_{ij}u_{kl}u_{ij}$$

and then, by Lemma 2.2.1, $u_{ij}$ and $u_{kl}$ commute. First, we note, that if $p = l$, we have $u_{kl}u_{il} = 0$, since $k \neq i$. Next, if $d(l,p) = 1$, we have $u_{kl}u_{ip} = 0$, since $d(k,i) = 2$, and then the statement follows from Lemma 2.2.3. The last case is thus $d(l,p) = 2$. We note, that by the Properties 4.4.2 of $\Gamma$, $l$ is part of exactly one quadrangle connecting all 4 triangles, i.e. there is only one vertex $v$, such that $d(l,v) = 2$ and $Q(l,v) = 4$. Since $j$ already fulfills both of these properties, we know that $Q(l,p) \neq 4$. Therefore, we have $Q(k,i) \neq Q(l,p)$ and thus $u_{kl}u_{ip} = 0$.

Step 6: We now consider $Q(i,k) = Q(j,l) = 2$. Let $p$ be again a vertex in $V \setminus \{j\}$. As above, we want to show

$$u_{ij}u_{kl}u_{ip} = 0.$$

Again, we already get $u_{kl}u_{ip} = 0$ if $d(l,p) \in \{0,1\}$. Let therefore $d(l,p) = 2$. If now $Q(l,p) \neq 2$, we get $u_{kl}u_{ip} = 0$ by Steps 2 and 4, as $Q(i,k) = 2$. We thus consider $Q(l,p) = 2$. Let now $q$ be a common neighbour of $i$ and $k$ such that $q \in T(k)$ and let $s$ and $t$ be the common neighbours of $j$ and $l$, such that $s \in T(j)$ and $t \in T(l)$. In particular, we have that $q \notin T(i)$. We denote by $\mathcal{Q}(j,l)$ the unique quadrangle containing $j$ and $l$ and by $\mathcal{Q}(l,p)$ the unique quadrangle containing $l$ and $p$. We now argue, that $d(t,p) \neq 1$. If $t$ and $p$ were connected, there would be a quadrangle consisting of vertices $l \sim t \sim p \sim v \sim l$, with the fourth vertex being a common neighbour $v \neq t$ of $l$ and $p$. Such a vertex $v$ exists since $Q(l,p) = 2$. Since we know by the Properties 4.4.2 of $\Gamma$, that two vertices in the same triangle only share exactly one quadrangle, we thus get a contradiction, since we already have $t \in \mathcal{Q}(j,l) = l \sim t \sim j \sim s \sim l$. Therefore, $d(t,p) \neq 1$. We compute

$$u_{ij}u_{kl}u_{ip} = u_{ij}\sum_{v\in V} u_{qv}u_{kl}u_{ip} = u_{ij}\sum_{\substack{v\in V\\ v\sim j\\ v\sim l}} u_{qv}u_{kl}u_{ip} = \underbrace{u_{ij}u_{qs}u_{kl}u_{ip}}_{\substack{=0 \text{ since } q\notin T(i)\\ \text{but } s\in T(j)}} + u_{ij}u_{qt}u_{kl}u_{ip}.$$

Since $q \in T(k)$ and $t \in T(l)$ holds however, we get by Step 1, that $u_{qt}$ and $u_{kl}$ commute, and we can continue the above calculation:

$$u_{ij}u_{qt}u_{kl}u_{ip} = u_{ij}u_{kl}u_{qt}u_{ip}.$$

However, $q$ is a common neighbour of $i$ and $k$, which means we have $d(i,q) = 1$, but as we argued above $d(t,p) \neq 1$, and thus, by Lemma 2.2.3, we have $u_{qt}u_{ip} = 0$ and thus

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{kl}u_{qt}u_{ip} = 0.$$

Therefore, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ by Lemma 2.2.1.

Step 7: Before we can go on to the case where $Q(i,k) = Q(j,l) = 0$, we need to consider the last case of $d(i,k) = d(j,l) = 1$. Let therefore now be $d(i,k) = d(j,l) = 1$ and $T(i) \neq T(k)$ and $T(j) \neq T(l)$. We will now show, that $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$. Observe, that for $d(l,p) \neq 1$, we already get the statement. Moreover, if $T(p) = T(l)$, then we have $u_{kl}u_{ip} = 0$ by Step 1, since $T(k) \neq T(i)$ and we are done.

Let therefore $T(p) \neq T(l)$. We observe, that there is a quadrangle, containing both $l$ and $j$, that connects 2 triangles. We denote by $q$ the vertex in this quadrangle, that is in the same triangle as $t$. In particular, we have $d(q,l) = 2$ and $Q(q,l) = 2$. Then, we have $d(p,q) \neq 1$: Assume $p \sim q$. Then $j \sim q \sim p \sim l \sim j$ would form a quadrangle. However, if $p$ were in $T(j)$, then $p$ would be a common neighbour of $j$ and $l$. Since such a common neighbour does not exist, we conclude $T(p) \neq T(j)$. Since we moreover have $T(j) = T(q)$, but $T(j) \neq T(l)$, $T(l) \neq T(p)$ and $T(j) \neq T(p)$, this quadrangle would connect 3 distinct triangles. Such a quadrangle does not exist in $\Gamma$ however, and we conclude $d(p,q) \neq 1$. We can now compute

$$u_{ij}u_{kl}u_{ip} = u_{ij} \sum_{\substack{v \in V \\ v \sim i \\ d(k,v)=2 \\ Q(k,v)=2}} u_{vq}u_{kl}u_{ip} = u_{ij}u_{kl} \underbrace{\sum_{\substack{v \in V \\ v \sim i \\ d(k,v)=2 \\ Q(k,v)=2}} u_{vq}u_{ip}}_{\substack{=0 \text{ since} \\ d(i,v)=1 \text{ but} \\ d(q,p)=2}} = 0.$$

Here, the sum commutes with $u_{kl}$ by Step 6, as $d(k,s) = 2 = d(q,l)$ and $Q(k,s) = 2 = Q(q,l)$ for all summands holds. We conclude

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$$

and therefore $u_{ij}$ and $u_{kl}$ commute by Lemma 2.2.1.

Step 8: Let now $Q(i,k) = Q(j,l) = 0$. We show again that

$$u_{ij}u_{kl}u_{ip} = 0$$

for all vertices $p \neq j$. As above, we already get that $u_{kl}u_{ip} = 0$, whenever $d(l,p) \neq d(i,k) = 2$ or when $d(l,p) = 2$ but $Q(l,p) \neq Q(i,k) = 0$. We thus now consider $p \in V \setminus \{j\}$ such that $d(l,p) = 2$ and $Q(l,p) = 0$. We will denote by $s$ the unique neighbour of $i$ and $k$, and by $t$ the unique neighbour of $l$ and $p$. We compute

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{kl} \sum_{v \in V} u_{vt}u_{ip} = u_{ij}u_{kl}u_{st}u_{ip}.$$

We moreover denote by $t'$ the unique neighbour of $j$ and $l$. Then in particular $t \neq t'$, since $p \neq j$. Similarly to above, we get

$$u_{ij}u_{kl}u_{st}u_{ip} = u_{ij}u_{st'}u_{kl}u_{st}u_{ip}.$$

Since however $d(s,k) = 1 = d(l,t')$, we know that $u_{st'}$ and $u_{kl}$ commute, and thus

$$u_{ij}u_{st'}u_{kl}u_{st}u_{ip} = u_{ij}u_{kl}u_{st'}u_{st}u_{ip} = 0$$

as $t \neq t'$. We thus have $u_{ij}u_{kl}u_{ip} = 0$ for $p \neq j$ and thus

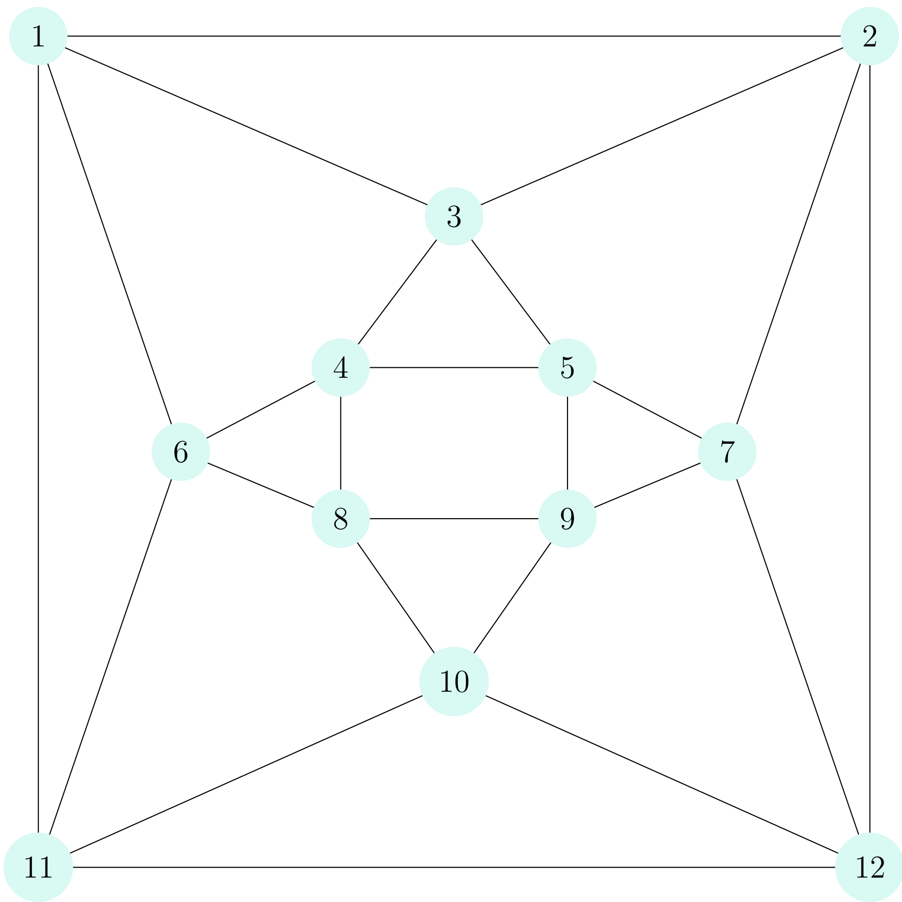
$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$$

and by Lemma 2.2.1, $u_{ij}$ and $u_{kl}$ commute.

□

**4.4.5. The cuboctahedral graph.** In this section, we show that the cuboctahedral graph, which is the line graph of the cube, has no quantum symmetries. By $\Gamma$ we will denote the cuboctahedral graph throughout the entire section, even if it is sometimes not explicitly stated.

4.4.1. Theorem. *Let $\Gamma$ be the cuboctahedral graph as shown below. It has no quantum symmetries, i.e. $C(G^+_{aut}(\Gamma))$ is commutative.*




Before proving the statement, we will again first collect some properties of the graph in question.

4.4.2. Properties.

- *$\Gamma$ has diameter 3, i.e. the largest distance between two vertices is 3.*
- *$\Gamma$ is 4-regular, i.e. every vertex has exactly 4 neighbours.*

- $\Gamma$ *contains* 8 *triangles.*
- *Each vertex of* $\Gamma$ *is contained in exactly two triangles and every pair of triangles overlaps in at most one vertex.*

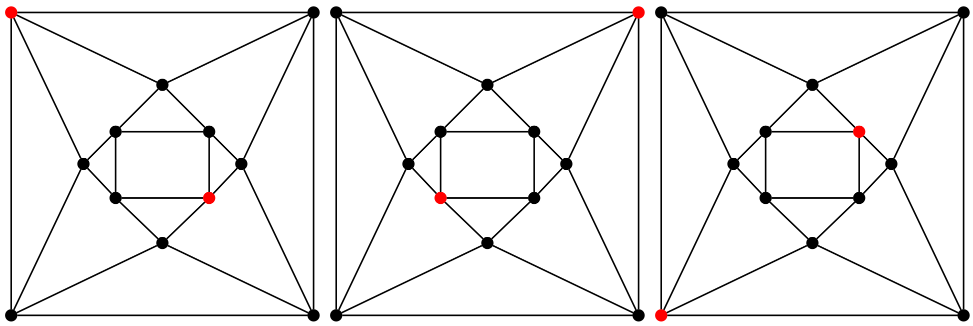
4.4.3. Lemma. *Let* $\Gamma = (V, E)$ *be the cuboctahedral graph. Then all vertices that are adjacent have exactly one common neighbour.*

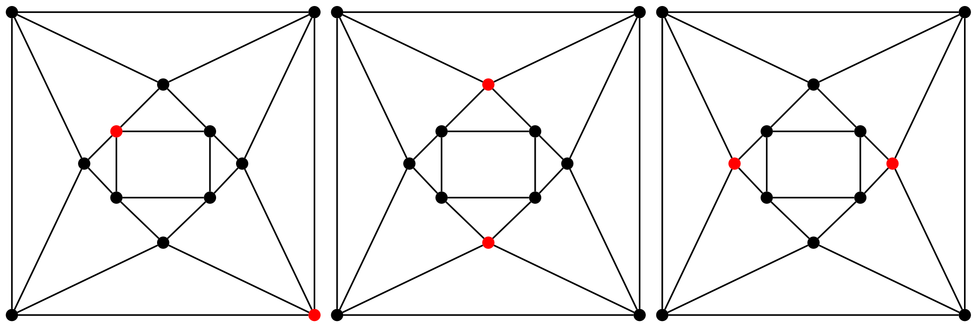
Proof. $\Gamma$ is the line graph of the cube $C = (V', E')$, i.e. the vertices of $\Gamma$ are the edges of $C$ and two vertices of $\Gamma$ are connected, if they share a vertex as edges of $C$. Let now $d, e \in V = E'$ be adjacent vertices in $\Gamma$, i.e. there is a vertex $v \in V'$ of $C$, such that $v \in d$ and $v \in e$. Since the cube is 3-regular, there is exactly one other edge, let us call it $f$, that contains $v$. Then $f$ is a common neighbour of $d$ and $e$ in $\Gamma$.

It remains to be shown, that $d$ and $e$ have no other common neighbour. For this, let $g \in E' \setminus \{d, e, f\}$ be another edge of $C$ and let us assume that $g$ is a common neighbour of $d$ and $e$ in $\Gamma$. It holds that $v \notin g$ since $C$ is 3-regular and the three edges containing $v$ are already $d$, $e$ and $f$. Let us therefore fix $g = (x, y)$ for $x, y \in V' \setminus \{v\}$. Since we have $(g, d) \in E$ and $(g, e) \in E$ by assumption, there is a vertex in $g$ for both $d$ and $e$ that they share with $g$. Since $(d, e) \in E$ we have $d \neq e$ and thus the vertices they share with $g$ are distinct. Thus, without loss of generality, we have $d = (v, x)$ and $e = (v, y)$. But then $\{x, y, v\}$ form a triangle. Since the cube does not contain any triangles, we have a contradiction, and thus $g$ can not be a common neighbour of $d$ and $e$. $\square$

4.4.4. Lemma. *For each vertex* $v$ *of* $\Gamma$*, there is exactly one vertex* $w$ *of* $\Gamma$*, such that* $d(v, w) = 3$*. We define the map* $a : V(\Gamma) \to V(\Gamma)$ *as the map mapping each vertex to that unique vertex in distance* 3*.*

Proof. We give the proof by giving illustrations of the 6 pairs of vertices in distance 3.





□

4.4.5. Lemma. *Let $v, w \in V$ be two vertices of $\Gamma$. If $d(v, w) = 2$ then there exists at least one vertex $x \in V$ that shares a triangle with both $v$ and $w$.*

Proof. Consider a shortest path from $v$ to $w$ and name the vertex that is passed by $x$. Since all edges that contain vertex $v$ only lead to vertices that share a triangle with $v$, $x$ shares a triangle with $v$. By the same argument, $x$ and $w$ share a triangle. □

4.4.6. Lemma. *Let $v \in V$ be a vertex in $\Gamma$. There are exactly two vertices that have distance 2 to both $v$ and $a(v)$. Moreover, if $w$ is one of these vertices, the other one is $a(w)$.*

Proof. We denote by $N(v)$ the neighbourhood of $v$, i.e. all vertices that are adjacent to $v$. Note, that $\#(N(v) \cup N(a(v))) = 8$, i.e. 8 vertices are either adjacent to $v$ or to $a(v)$. This is due to the fact that $\Gamma$ has degree 4 and thus all vertices have 4 neighbours and moreover $N(v) \cap N(a(v)) = \{\}$. This holds, since if there was a vertex adjacent to both $v$ and $a(v)$, this would yield a path from $v$ to $a(v)$ of length 2, which is a contradiction to $d(v, a(v)) = 3$. We thus have

$$V \backslash (N(v) \cup N(a(v)) \cup \{v, a(v)\}) = \{w, x\}$$

for some vertices $w, x$. One can check by hand, that $d(w, x) = 3$ for all choices of $v$. □

We now come to the proof of Theorem 4.4.1.

Proof. of Theorem 4.4.1 Let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. We consider again vertices $i, j, k, l$ in increasing order of $m = d(i, k) = d(j, l)$. Since $\Gamma$ has diameter 3, $m$ is maximally 3.

Step 1: $m = 1$. Since we know by Lemma 4.4.3 that in $\Gamma$ all adjacent vertices have exactly one common neighbour, we can conclude by Lemma 2.2.10 that $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ and therefore $u_{ij}u_{kl} = u_{kl}u_{ij}$.

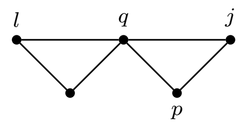
Step 2: $m = 2$. We consider two cases:

`Step 2.1:` $d(a(j),l) = 1$. If the unique vertex at distance 3 from $j$ is adjacent to $l$, we can apply Lemma 2.2.4 by putting $q := a(j)$ and get

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(l,p)=2\\ d(a(j),p)=3}} u_{ip} = u_{ij}u_{kl}u_{ij}$$

since $j$ is the only vertex at distance 3 to $a(j)$. We can apply the lemma, since we have $G^+_{aut}(\Gamma) = G^*_{aut}(\Gamma)$ by `Step 1`. We thus can apply Lemma 2.2.1 to see that $u_{ij}$ and $u_{kl}$ commute.

`Step 2.2:` $d(a(j),l) = 2$. We apply again Lemma 2.2.4 and this time we put $q$ as a vertex that shares a triangle with both $j$ and $l$, which exists by Lemma 4.4.5. Let us give a sketch of the situation.




This yields

$$u_{ij}u_{kl} = u_{ij}u_{kl} \sum_{\substack{p;d(p,l)=2\\ (p,q)\in E}} u_{ip}.$$

The vertex $q$ has 4 neighbours, two of which are in the same triangle as $l$ and thus have distance smaller than 2 to $l$. The remaining vertices are $j$ and $p$ and we thus get

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{ip}).$$

We now want to show, that $d(a(p),l) = 1$. For this, first note, that by Lemma 4.4.6 there are exactly 2 vertices that have distance 2 to both $j$ and $a(j)$ and since $l$ is one of those, the other one is $a(l)$. Moreover, this yields that the only vertices at distance 2 to both $l$ and $a(l)$ are in turn $j$ and $a(j)$. Therefore we get

$$x \notin \{j, a(j), l, a(l)\} \text{ it holds that } d(x,l) = 1 \text{ or } d(x,a(l)) = 1. \tag{4.4.2}$$

Since we have that $d(p,j) = 1$, and thus $p \neq j$, and $d(p,l) = 2$ we thus get $d(p,a(l)) = 1$. But then we get $d(a(p),a(l)) > 1$, since otherwise there would be a path from $p$ to $a(p)$ of length 2 via $a(l)$, which can not be. But then we get, again applying 4.4.2, that $d(a(p),l) = 1$. Now, we can use our results from `Step 2.1` to see that $u_{kl}$ and $u_{ip}$ commute. This yields

$$u_{ij}u_{kl}u_{ip} = u_{ij}u_{ip}u_{kl} = 0$$

since $j \neq p$. Therefore

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$$

and again by Lemma 2.2.1, $u_{ij}$ and $u_{kl}$ commute.

Step 3: $m = 3$. If $d(i,k) = d(j,l) = 3$ holds, we get with Lemma 2.2.3 that

$$u_{ij}u_{kl} = u_{ij}u_{kl}\sum_{p\in V} u_{ip} = u_{ij}u_{kl}\sum_{p;d(l,p)=3} u_{ip}.$$

But since the only vertex at distance 3 from $l$ is $j$ this yields

$$u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$$

and by Lemma 2.2.1, $u_{ij}$ and $u_{kl}$ commute.

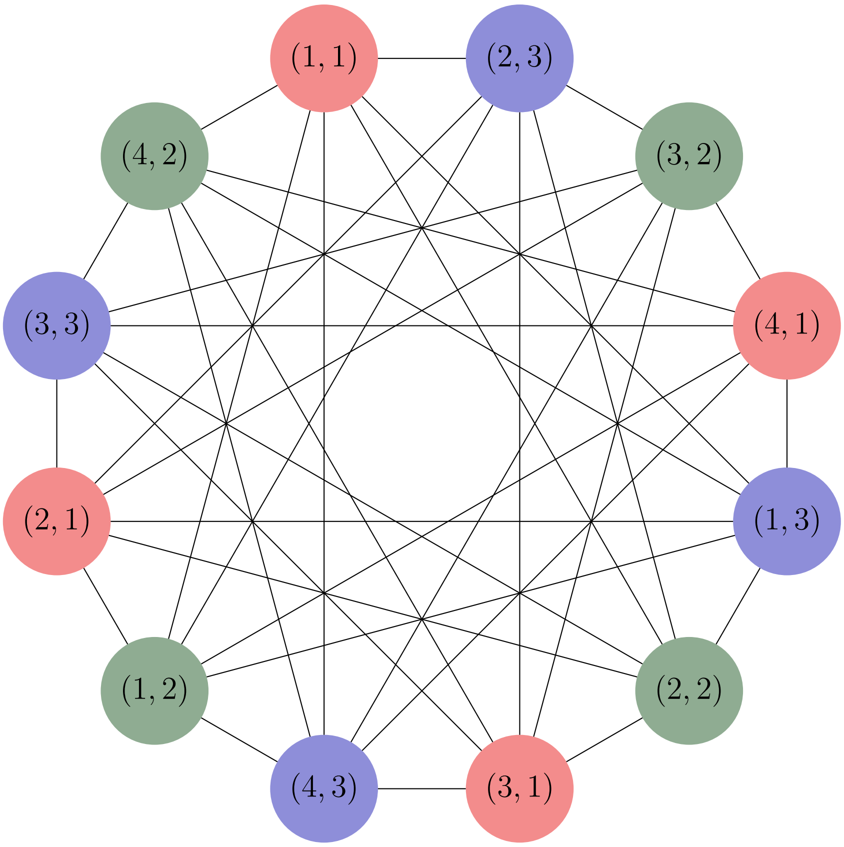
□

### 4.5. Computation of Quantum Automorphism Groups

In this section, we give the computation of the quantum automorphism groups of $C_{12}(4,5)$ and $C_{12}(3^+,6)$. For these computations, we used in some places a non-commutative Buchberger implementation as described in Section 3.1, which was implemented in the `OSCAR` [66] package in the `julia` programming language [12].

4.5.1. Theorem. *The quantum automorphism group of $C_{12}(4,5)$ is $H_2^+ \times S_3$.*

Proof. For the purpose of this proof, we will call $C_{12}(4,5) =: \Gamma$. We first give an illustration of the graph that highlights the action of the automorphism group on its vertices, which will help in constructing the isomorphism between $G^+_{aut}(\Gamma)$ and $H_2^+ \times S_3$. We also label the vertices of this graph differently than before to make it easier to talk about the construction of the $*$-isomorphism between the two quantum groups. Note that the colors in this illustration are only to make it easier to see the action of the automorphism group on $\Gamma$ and are not an actual graph coloring.




The universal $C^*-$algebra defining $H_2^+ \times S_3$ is as follows:

$$
\begin{aligned}
C(H_2^+ \times S_3) = C^*(v_{ij}, w_{kl}, 1 \leq i,j \leq 4, 1 \leq k,l \leq 3 \\
&\sum_{a=1}^{4} v_{ia} = \sum_{a=1}^{4} v_{aj} = 1 \ \forall 1 \leq i,j \leq 4 \\
&\sum_{b=1}^{3} w_{kb} = \sum_{b=1}^{3} w_{bl} = 1 \ \forall 1 \leq k,l \leq 3 \\
&v_{ij}^2 = v_{ij} = v_{ij}^* \\
&w_{ij}^2 = w_{ij} = w_{ij}^* \\
&A_{C_4} v = v A_{C_4} \\
&v_{ij} w_{kl} = w_{kl} v_{ij}).
\end{aligned}
$$

Here, $A_{C_4}$ is the adjacency matrix of the 4-cycle $C_4$ and $v$ is the matrix with entries $v_{ij}$ for $1 \leq i,j \leq 4$. We use the fact that $G_{aut}^+(C_4) = H_2^+$, which was shown in [76].

We now claim:

(i) The matrix $\hat{u}$ with entries from $C(H_2^+ \times S_3)$ defined by

$$\hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}$$

satisfies the relations of $C(G_{aut}^+(\Gamma))$.

(ii) The elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ in $C(G^+_{aut}(\Gamma))$ defined by

$$\hat{v}_{ij} = u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)}$$
$$\hat{w}_{kl} = u_{(1,k)(1,l)} + u_{(1,k)(2,l)} + u_{(1,k)(3,l)} + u_{(1,k)(4,l)}$$

satisfy the relations of $C(H_2^+ \times S_3)$.

Let us first prove (i):

For this, observe the following: We have

$$(i_1, i_2) \sim (k_1, k_2) \text{ in } \Gamma$$
$$\iff i_2 \neq k_2 \wedge (i_1 = k_1 \vee i_1 \sim_{C_4} k_1)$$

Let now $(i_1, i_2) \sim (k_1, k_2)$ and $(j_1, j_2) \not\sim (l_1, l_2)$, i.e. for $(i_1, i_2)$ and $(k_1, k_2)$ the above observation holds, while for $(j_1, j_2)$ and $(l_1, l_2)$ we have:

$$j_2 = l_2 \vee (j_1 \neq l_1 \wedge j_1 \not\sim_{C_4} l_1).$$

In order to show that indeed $\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0$, we make the following case distinction:

- Case $j_2 = l_2$:
  Since we already know $i_2 \neq k_2$, we get $w_{i_2j_2}w_{k_2j_2} = 0$ and thus
  $\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = v_{i_1j_1}w_{i_2j_2}v_{k_1l_1}w_{k_2j_2} = v_{i_1j_1}w_{i_2j_2}w_{k_2j_2}v_{k_1l_1} = 0$
- Case $j_2 \neq l_2$:
  - Case $i_1 = k_1$:
    We get $v_{i_1j_1}v_{i_1l_1} = 0$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0$
  - Case $i_1 \sim_{C_4} k_1$ :
    Since $j_1 \not\sim_{C_4} l_1$, we get $v_{i_1j_1}v_{k_1l_1} = 0$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)}\hat{u}_{(k_1,k_2)(l_1,l_2)} = 0$

It is moreover easy to see, that the relations of $H_2^+ \times S_3$ yield that all rows and columns of $\hat{u}$ sum up to 1 and that the entries of $\hat{u}$ are projections, as they are just products of commuting projections. From the universal property of $C(G^+_{aut}(\Gamma))$ we thus get a $*$-homomorphism

$$\varphi : C(G^+_{aut}(\Gamma)) \to C(H_2^+ \times S_3)$$
$$u_{(i_1,i_2)(j_1,j_2)} \mapsto \hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1j_1}w_{i_2j_2}.$$

It is also easy to see that all the generators of $C(H_2^+ \times S_3)$ are in the image of $\varphi$: for example, $v_{i_1j_1}$ is the image of

$$u_{(i_1,1)(j_1,1)} + u_{(i_1,1)(j_1,2)} + u_{(i_1,1)(j_1,3)}$$

since

$$w_{11} + w_{12} + w_{13} = 1,$$

and a similar argument holds for $w_{i_2j_2}$. Therefore, $\varphi$ is surjective.

We will next prove (ii):

We first note that using the Gröbner basis for the ideal generated by the relations of $C(G^+_{aut}(\Gamma))$, we see that

$$u_{(i,a)(k,b)} = u_{(i',a)(k',b)} \text{ for } i \not\sim_{C_4} i',\ i \neq i' \text{ and } k \not\sim_{C_4} k',\ k \neq k'. \tag{4.5.1}$$

Let $i \not\sim_{C_4} k$ and $j \sim_{C_4} l$ be given. We then have $(i,1) \not\sim (k,1)$ but $(j,a) \sim (l,b)$ whenever $a \neq b$. Therefore, $u_{(i,1)(j,a)}u_{(k,1)(l,b)} = 0$ whenever $a \neq b$ and thus in the product

$$\hat{v}_{ij}\hat{v}_{kl} = \big(u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)}\big)\big(u_{(k,1)(l,1)} + u_{(k,1)(l,2)} + u_{(k,1)(l,3)}\big)$$

we only keep those products, where $a = b$ and are left with

$$\hat{v}_{ij}\hat{v}_{kl} = u_{(i,1)(j,1)}u_{(k,1)(l,1)} + u_{(i,1)(j,2)}u_{(k,1)(l,2)} + u_{(i,1)(j,3)}u_{(k,1)(l,3)}.$$

By (4.5.1), we know that $u_{(k,1)(l,a)} = u_{(i,1)(l',a)}$, where $l' \not\sim_{C_4} l$. Since $j \sim_{C_4} l$, we know in particular that $l' \neq j$ and get $u_{(i,1)(j,a)}u_{(i,1)(l',a)} = 0$ and thus

$$\hat{v}_{ij}\hat{v}_{kl} = 0.$$

Let now $i \sim_{C_4} k$ and $j \not\sim_{C_4} l$ be given. Using a similar trick to above, we see that $u_{(i,1)(j,a)}u_{(k,1)(l,a)} = 0$, only this time using the fact that $u_{(k,1)(l,a)} = u_{(k',1)(j,a)}$, and get that all summands of that form in $\hat{v}_{ij}\hat{v}_{kl}$ disappear to get

$$\begin{aligned}\hat{v}_{ij}\hat{v}_{kl} =& u_{(i,1)(j,1)}u_{(k,1)(l,2)} + u_{(i,1)(j,1)}u_{(k,1)(l,3)}\\ &+u_{(i,1)(j,2)}u_{(k,1)(l,1)} + u_{(i,1)(j,2)}u_{(k,1)(l,3)}\\ &+u_{(i,1)(j,3)}u_{(k,1)(l,1)} + u_{(i,1)(j,2)}u_{(k,1)(l,2)}.\end{aligned}$$

Using again (4.5.1) to see that $u_{(k,1)(l,a)} = u_{(k',1)(j,a)}$, we get

$$\begin{aligned}\hat{v}_{ij}\hat{v}_{kl} =& u_{(i,1)(j,1)}u_{(k',1)(j,2)} + u_{(i,1)(j,1)}u_{(k',1)(j,3)}\\ &+u_{(i,1)(j,2)}u_{(k',1)(j,1)} + u_{(i,1)(j,2)}u_{(k',1)(j,3)}\\ &+u_{(i,1)(j,3)}u_{(k',1)(j,1)} + u_{(i,1)(j,2)}u_{(k',1)(j,2)}.\end{aligned}$$

However, we know, that still $(i,1) \not\sim (k',1)$ holds but we also have $(j,a) \sim (j,b)$ for $a \neq b$. Therefore, we have $u_{(i,1)(j,a)}u_{(k',1)(j,b)} = 0$ and get

$$\hat{v}_{ij}\hat{v}_{kl} = 0.$$

It is easy to see, that the $\hat{v}_{(ij)}$ and the $\hat{w}_{kl}$ are self-adjoint, as they are just sums of self-adjoint elements. Moreover, since $u_{(i_1,i_2)(j_1,j_2)}u_{(i_1,i_2)(l_1,l_2)} = 0$ for $(j_1,j_2) \neq (l_1,l_2)$, seeing $\hat{v}_{ij}^2 = \hat{v}_{ij}$ and $\hat{w}_{kl}^2 = \hat{w}_{kl}$ is also straightforward.

It just remains to be shown that $\sum_a \hat{v}_{ia} = \sum_a \hat{v}_{aj} = 1 = \sum_b \hat{w}_{kb} = \sum_b \hat{w}_{bl}$. This is easy to see for $\sum_a \hat{v}_{ia}$ and $\sum_b \hat{w}_{bl}$, since these sums are just sums over one row of $u$. For $\sum_a \hat{aj} = 1$ and $\sum_b \hat{w}_{kb}$, the Gröbner basis yields that the sums are equal to 1.

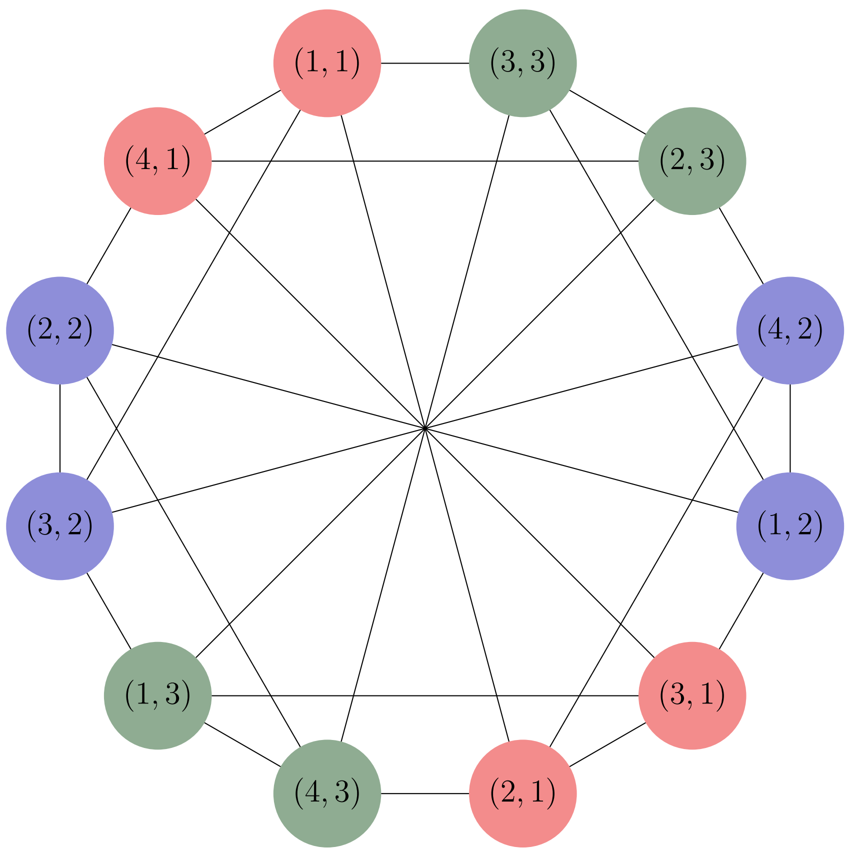
All in all, we get again by the universal property of $C(H_2^+ \times S_3)$ a $*$-homomorphism

$$\varphi' : C(H_2^+ \times S_3) \to C(G^+_{aut}(\Gamma))$$
$$v_{ij} \mapsto \hat{v}_{ij}$$
$$w_{ij} \mapsto \hat{w}_{ij}.$$

By (4.5.1), we see that all elements of $u$ are hit by $\varphi'$ and in particular, we have that $\varphi'$ is the inverse of $\varphi$. We thus get, that $C(H_2^+ \times S_3)$ and $C(G^+_{aut}(\Gamma))$ are $*$-isomorphic. $\square$

4.5.2. Theorem. *The quantum automorphism group of $C_{12}(3^+, 6)$ is $H_2^+ \times S_3$.*

Proof. For this proof, we will write $C_{12}(3^+, 6) =: \Gamma$. We will give a picture of the graph in question with vertices colored in such a way, that the action of the classical automorphism group $H_2 \times S_3$ is visible:



Recall from above that he universal $C^*$−algebra defining $H_2^+ \times S_3$ is as follows:

$$\begin{aligned} C(H_2^+ \times S_3) = C^*(v_{ij}, w_{kl}, 1 \leq i,j \leq 4, 1 \leq k,l \leq 3 \\ &\sum_{a=1}^{4} v_{ia} = \sum_{a=1}^{4} v_{aj} = 1\ \forall 1 \leq i,j \leq 4 \\ &\sum_{b=1}^{3} w_{kb} = \sum_{b=1}^{3} w_{bl} = 1\ \forall 1 \leq k,l \leq 3 \\ &v_{ij}^2 = v_{ij} = v_{ij}^* \\ &w_{ij}^2 = w_{ij} = w_{ij}^* \\ &A_{C_4} v = v A_{C_4} \\ &v_{ij} w_{kl} = w_{kl} v_{ij}). \end{aligned}$$

Here, $A_{C_4}$ is the adjacency matrix of the 4-cycle $C_4$ and $v$ is the matrix with entries $v_{ij}$ for $1 \leq i,j \leq 4$.

We now claim:

(i) The matrix $\hat{u}$ with entries from $C(H_2^+ \times S_3)$ defined by

$$\hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1 j_1} w_{i_2 j_2}$$

satisfies the relations of $C(G_{aut}^+(\Gamma))$.

(ii) The elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ in $C(G_{aut}^+(\Gamma))$ defined by

$$\begin{aligned} \hat{v}_{ij} &= u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)} \\ \hat{w}_{kl} &= u_{(1,k)(1,l)} + u_{(1,k)(2,l)} + u_{(1,k)(3,l)} + u_{(1,k)(4,l)} \end{aligned}$$

satisfy the relations of $C(H_2^+ \times S_3)$.

Let us first prove (i):

For this, observe the following: We have

$$\begin{aligned} &(i_1,i_2) \sim (k_1,k_2) \text{ in } \Gamma \\ \iff &(i_2 = k_2 \wedge i_1 \sim_{C_4} k_1) \vee (i_2 \neq k_2 \wedge i_1 \neq k_1 \wedge i_1 \not\sim_{C_4} k_1) \end{aligned}$$

Let now $(i_1,i_2) \sim (k_1,k_2)$ and $(j_1,j_2) \not\sim (l_1,l_2)$, i.e. for $(i_1,i_2)$ and $(k_1,k_2)$ the above observation holds, while for $(j_1,j_2)$ and $(l_1,l_2)$ we have:

$$(j_2 \neq l_2 \vee j_1 \not\sim_{C_4} l_1) \wedge (j_2 = l_2 \vee j_1 = l_1 \vee j_1 \sim_{C_4} l_1).$$

In order to show that $\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0$, we will make the following case distinction:

- Case $i_2 = k_2$:
  - Case $j_2 \neq l_2$:
    We have $\hat{u}_{(i_1,i_2)(j_1,j_2)} \hat{u}_{(k_1,k_2)(l_1,l_2)} = v_{i_1 j_1} w_{i_2 j_2} v_{k_1 l_1} w_{i_2 l_2} = v_{i_1 j_1} w_{i_2 j_2} w_{i_2 l_2} v_{k_1 l_1}$ but $w_{i_2 j_2} w_{i_2 l_2} = 0$ since $j_2 \neq l_2$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)} u_{(k_1,k_2)(l_1,l_2)} = 0$.

- Case $j_2 = l_2$:
  We have $i_1 \sim_{C_4} k_1$ but $j_1 \not\sim_{C_4} l_1$ and thus $v_{i_1j_1}v_{k_1l_1} = 0$ and therefore also $\hat{u}_{(i_1,i_2)(j_1,j_2)}u_{(k_1,k_2)(l_1,l_2)} = 0$.

- Case $i_2 \neq k_2$:
  - Case $j_2 = l_2$:
    We have $w_{i_2j_2}w_{k_2j_2} = 0$ since $i_2 \neq k_2$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)}u_{(k_1,k_2)(l_1,l_2)} = 0$.
  - Case $j_2 \neq l_2, j_1 = l_1$:
    We have $v_{i_1j_1}v_{k_1j_1} = 0$ since $i_1 \neq k_1$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)}u_{(k_1,k_2)(l_1,l_2)} = 0$.
  - Case $j_2 \neq l_2, j_1 \sim_{C_4} l_1$:
    We have $v_{i_1j_1}v_{k_1l_1} = 0$ since $i_1 \not\sim_{C_4} k_1$ but $j_1 \sim_{C_4} l_1$ and thus $\hat{u}_{(i_1,i_2)(j_1,j_2)}u_{(k_1,k_2)(l_1,l_2)} = 0$.

A similar computation will show, that $\hat{u}_{(i_1,i_2)(j_1,j_2)}u_{(k_1,k_2)(l_1,l_2)} = 0$ if $(i_1,i_2) \not\sim (k_1,k_2)$ and $(j_1,j_2) \sim (l_1,l_2)$ holds.

It is immediate to see, that $\hat{u}_{(i_1,i_2)(j_1,j_2)}$ is a projection again and that the rows and columns of $\hat{u}$ sum up to one from the relations of $C(H_2^+ \times S_3)$. We thus have that $\hat{u}$ satisfies the relations of $C(G_{aut}^+(\Gamma))$ and from the universal property of $C(G_{aut}^+(\Gamma))$ we get a $*$-homomorphism

$$\varphi : C(G_{aut}^+(\Gamma)) \to C(H_2^+ \times S_3)$$
$$u_{(i_1,i_2)(j_1,j_2)} \mapsto \hat{u}_{(i_1,i_2)(j_1,j_2)} = v_{i_1j_1}w_{i_2j_2}.$$

It is also easy to see that all the generators of $C(H_2^+ \times S_3)$ are in the image of $\varphi$: for example, $v_{i_1j_1}$ is the image of

$$u_{(i_1,1)(j_1,1)} + u_{(i_1,1)(j_1,2)} + u_{(i_1,1)(j_1,3)}$$

since

$$w_{11} + w_{12} + w_{13} = 1,$$

and a similar argument holds for $w_{i_2j_2}$. Therefore, $\varphi$ is surjective.

We will now prove (ii).

Firstly, using the Gröbner basis for the ideal generated by the relations of $C(G_{aut}^+(\Gamma))$, we can see that

$$\hat{v}_{i_1j_1}\hat{w}_{i_2j_2} = u_{(i_1,i_2)(j_1,j_2)} = \hat{w}_{i_2j_2}\hat{v}_{i_1j_1}. \tag{4.5.2}$$

In particular, we see that all the elements $\hat{v}_{ij}$ commute with all the $\hat{w}_{kl}$ as desired. It is moreover easy to see, that $\hat{v}_{ij}^* = \hat{v}_{ij}$ and $\hat{w}_{kl}^* = \hat{w}_{kl}$ since all the entries of $u$ are already selfadjoint.

Next, we compute

$$\begin{aligned}\hat{v}_{ij}^2 =&(u_{(i,1)(j,1)} + u_{(i,1)(j,2)} + u_{(i,1)(j,3)})^2\\ =&u_{(i,1)(j,1)} + \underbrace{u_{(i,1)(j,1)}u_{(i,1)(j,2)} + u_{(i,1)(j,1)}u_{(i,1)(j,3)}}_{=0} +\\ &u_{(i,1)(j,2)} + \underbrace{u_{(i,1)(j,2)}u_{(i,1)(j,1)} + u_{(i,1)(j,2)}u_{(i,1)(j,3)}}_{=0} +\\ &u_{(i,1)(j,3)} + \underbrace{u_{(i,1)(j,3)}u_{(i,1)(j,1)} + u_{(i,1)(j,3)}u_{(i,1)(j,2)}}_{=0}\\ =&\hat{v}_{ij}.\end{aligned}$$

A similar computation can be done to see that $\hat{w}_{kl}^2 = \hat{w}_{kl}$, and thus all elements $\hat{v}_{ij}$ and $\hat{w}_{kl}$ are projections.

Next, it is easy to see that

$$\sum_{a=1}^{4} \hat{v}_{ia} = \sum_{a=1}^{4}\sum_{j=1}^{3} u_{(i,1)(a,j)} = 1$$
$$\sum_{b=1}^{3} \hat{w}_{kb} = \sum_{b=1}^{3}\sum_{l=1}^{4} u_{(k,1)(b,l)} = 1.$$

Using again the Gröbner basis, we see that also

$$\sum_{a=1}^{4} \hat{v}_{ai} = 1 = \sum_{b=1}^{3} \hat{w_{kb}}$$

holds.

Lastly, we now need to check, that $\hat{v}$ satisfies

$$A_{C_4}v = vA_{C_4},$$

in other words, we need to show that

$$\hat{v}_{ij}\hat{v}_{kl} = 0 \text{ if } i \sim_{C_4} k,\ j \not\sim_{C_4} l \text{ or } i \not\sim_{C_4} k\ j \sim_{C_4} l.$$

Let therefore $i \sim_{C_4} k$ and $j \not\sim_{C_4} l$ be given. Using the identity in (4.5.2), we write

$$\hat{v}_{ij} = \hat{v}_{ij}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3}$$

and compute

$$\begin{aligned}\hat{v}_{ij}\hat{v}_{kl} =&(\hat{v}_{ij}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3})(\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{kl}\hat{w}_{1,3})\\ =&\hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,3}+\\ &\hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,2}\hat{v}_{kl}\hat{w}_{1,3}+\\ &\hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,2} + \hat{v}_{ij}\hat{w}_{1,3}\hat{v}_{kl}\hat{w}_{1,3}.\end{aligned}$$

We have by (4.5.2), that $\hat{v}_{kl}$ and $\hat{w}_{ab}$ commute. Moreover, we have seen above that $\hat{w}_{ab}$ and $\hat{w}_{ac}$ are projections that sum up to one and thus $\hat{w}_{ab}\hat{w}_{ac} = 0$ for $b \neq c$. We thus get that any of the products in the above sum where both $\hat{w}_{1b}$ and $\hat{w}_{1c}$ appear for $b \neq c$ are 0 and we are only left with

$$\hat{v}_{ij}\hat{v}_{kl} = \hat{v}_{ij}\hat{w}_{1,1}\hat{v}_{kl}\hat{w}_{1,1} + \hat{v}_{ij}\hat{w}_{2,2}\hat{v}_{kl}\hat{w}_{2,2} + \hat{v}_{ij}\hat{w}_{3,3}\hat{v}_{kl}\hat{w}_{3,3}.$$

Using again (4.5.2), we rewrite this to

$$\hat{v}_{ij}\hat{v}_{kl} = u_{(i,1)(j,1)}u_{(k,1)(l,1)} + u_{(i,1)(j,2)}u_{(k,1)(l,2)} + u_{(i,1)(j,3)}u_{(k,1)(l,3)}.$$

However, since by assumption $i \sim_{C_4} k$ but $j \not\sim_{C_4} l$, we have $(i,1) \sim (k,1)$ but $(j,a) \not\sim (l,a)$ for any $1 \leq a \leq 3$. We thus get

$$u_{(i,1)(j,a)}u_{(k,1)(l,a)} = 0$$

and thus

$$\hat{v}_{ij}\hat{v}_{kl} = 0.$$

A similar computation yields the same result whenever $i \not\sim_{C_4} k$ and $j \sim_{C_4} l$ holds. All in all, the universal property of $C(H_2^+ \times S_3)$ yields again a $*$-homomorphism

$$\varphi' : C(H_2^+ \times S_3) \to C(G^+_{aut}(\Gamma))$$
$$v_{i_1 j_1} w_{i_2 j_2} \mapsto u_{(i_1,i_2)(j_1,j_2)}.$$

It is easy to see, that all generators $u_{(i_1,i_2)(j_1,j_2)}$ of $C(G^+_{aut}(\Gamma))$ are in the image and thus $\varphi'$ is surjective. We note moreover, that $\varphi'$ is exactly the inverse of $\varphi$ and we thus get that $H_2^+ \times S_3$ and $G^+_{aut}(\Gamma)$ are $*$-isomorphic. $\square$

CHAPTER 5

# Quantum Switching Isomorphism

The study of quantum symmetry has been done for many different objects, such as of graphs in [4], [13], hadamard matrices in [42] and hypergraphs in [36]. Closely related to this is the study of quantum isomorphisms, which has been defined for graphs in [2], where a nonlocal game was introduced that captures the notions of classical and quantum graph isomorphism in the following sense:

If $\Gamma_1$ and $\Gamma_2$ are finite, simple graphs with adjacency matrices $A_1$ and $A_2$ respectively, then we have

- There is a perfect classical strategy for the isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there exists a permutation matrix $P$ such that $A_1P = PA_2$.
- There is a perfect quantum strategy for the isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there exists a quantum permutation matrix $u$ such that $A_1u = uA_2$.

In a similar manner, we construct a nonlocal game that captures the notion of switching isomorphism for signed graphs:

If $\Gamma_1$ and $\Gamma_2$ are signed graphs with adjacency matrices $A_1$ and $A_2$, then we have:

- There is a perfect classical strategy for the switching isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there is a signed permutation matrix $H \in H_n$ such that $A_1H = HA_2$.
- There is a perfect quantum strategy for the switching isomorphism game of $\Gamma_1$ and $\Gamma_2$ if and only if there is a quantum signed permutation matrix $v$ such that $A_1v = vA_2$.

In the analysis of this new *quantum switching isomorphism* we note that for connected signed graphs any quantum switching isomorphism must come from a quantum isomorphism of the unsigned versions of the graphs. For non-connected graphs however, we did not find a similar result.

## 5.1. Basic Definitions

We now give some basic definitions surrounding signed graphs and their switching isomorphisms.

5.1.1. Definition. An *finite signed graph without multiple edges* $\Sigma$ is a finite graph $(V(\Sigma), E(\Sigma))$ together with a label function $\ell\colon E(\Sigma) \to \{\pm 1\}$. It is called *undirected*, if the base graph is undirected and if we have $\ell(u, v) = \ell(v, u)$ for all edges $(u, v) \in E(\Sigma)$.

The *adjacency matrix* $A_\Gamma$ of a signed graph on $n$ vertices without multiple edges is the $n \times n$-matrix with the $(i, j)$-entry being the value of the label function $\ell(i, j)$ if $i \sim j$ and 0 otherwise. If $\Gamma$ is undirected and without multiple edges, $A_\Gamma$ is thus a symmetric matrix with $\{0, \pm 1\}$-entries.

In this work, we only consider undirected finite signed graphs without multiple edges and without loops.

5.1.2. DEFINITION. Let $\Sigma_1$ and $\Sigma_2$ be two finite, simple signed graphs. Let the label functions be given by $\ell_1 \colon E(\Sigma_1) \to \{\pm 1\}$ and $\ell_2 \colon E(\Sigma_2) \to \{\pm 1\}$. We say $\Sigma_1$ and $\Sigma_2$ are *isomorphic* if there exists an isomorphism $\sigma$ of the base graphs and additionally

$$\ell_1(i, j) = \ell_2(\sigma(i), \sigma(j))$$

for any adjacent vertices $i, j \in V(\Sigma_1)$.

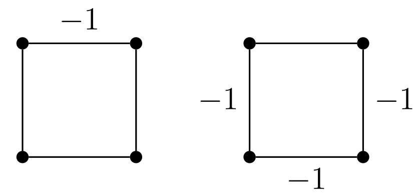
We say $\Sigma_1$ and $\Sigma_2$ are *switching isomorphic*, $\Sigma_1 \cong_{sw.} \Sigma_2$, if there is an isomorphism $\sigma \colon V(\Sigma_1) \to V(\Sigma_2)$ of the base graphs and a *switching function* $s \colon V(\Sigma_1) \to \{\pm 1\}$ such that

$$\ell_1(i, j) = s(i) \cdot s(j) \cdot \ell_2(\sigma(i), \sigma(j))$$

for any adjacent vertices $i, j \in V(\Sigma_1)$.

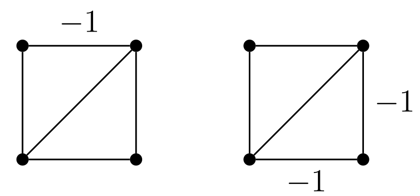
5.1.3. EXAMPLE.

- The following two signed graphs are not isomorphic, however they are switching isomorphic.




  Here, any edge without a label is implied to have the label $+1$.

- On the other hand, the following two graphs are neither isomorphic nor switching isomorphic.




The following alternative characterisation of switching isomorphism is sometimes also useful:

5.1.4. LEMMA. *Let $\Sigma_1$ and $\Sigma_2$ be finite, simple signed graphs on $n$ vertices. Let their adjacency matrices be given by $A_1$, $A_2 \in M_n(\{0, \pm 1\})$, such that $\ell_t(i, j) = A_t[i, j]$ for adjacent vertices $i, j$ from $\Gamma_t$, $t \in \{1, 2\}$. Then $\Sigma_1$ and $\Sigma_2$ are switching isomorphic as in Definition 5.1.2 if and only if there exists $H \in H_n$ such that*

$$A_1 H = H A_2.$$

PROOF. It is easy to see, that given a permutation $\sigma$ and a switching function $s$, a matrix $H$ of the form

$$H = \left(\delta_{\sigma(i)j} \cdot s(i)\right)_{ij}$$

is in $H_n$.

On the other hand, any matrix $H$ in $H_n$ can be written as above for a permutation $\sigma \in S_n$ and a function $s\colon \{1, \dots, n\} \to \{\pm 1\}$ and thus defines a switching isomorphism of the two graphs, since $A_1 H = H A_2$ implies

$$(A_1)_{ij} = \left(HA_2H^{-1}\right)_{ij} = H_{i\sigma(i)} H_{j\sigma(j)} (A_2)_{\sigma(i)\sigma(j)} = s(i)s(j)(A_2)_{\sigma(i)\sigma(j)}.$$

□

## 5.2. Construction of the Game

For this chapter, we will always assume that $\Gamma_1$ and $\Gamma_2$ are finite, simple signed graphs on $n$ vertices with adjacency matrices $A_1$ and $A_2$.

5.2.1. DEFINITION. Let $\Gamma_1$ and $\Gamma_2$ be two signed graphs on $n$ vertices. We identify their vertex sets $V_1 = V_2 =: V$. Let their label functions be given by $\ell_1\colon E_1 \to \{+1, -1\}$ and $\ell_2\colon E_2 \to \{+1, -1\}$. We then define the switching isomorphism game $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$ as follows:

The referee gives to both Alice and Bob a uniformly sampled vertex $x_A$ and $x_B$ from $\Gamma_1$. Alice and Bob each return a vertex from $\Gamma_2$ and a number from $\{+1, -1\}$. We call the return values $(y_A, s_A)$ and $(y_B, s_B)$ respectively. Alice and Bob win the game, if the following conditions are met:

(i) $x_A \sim x_B \iff y_A \sim y_B$
(ii) $x_A = x_B \iff y_A = y_B$
(iii) $x_A \sim x_B \implies \ell_1(x_A, x_B) = s_A s_B \ell_2(y_A, y_B)$
(iv) $x_A = x_B \implies s_A = s_B$

In the following, we will say that $\mathrm{rel}(x_A, x_B) = \mathrm{rel}(y_A, y_B)$ for the first two conditions.

We first show that this game captures the classical switching isomorphisms.

5.2.2. LEMMA. *There exists a perfect classical strategy for* $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$ *if and only if there exists a matrix* $H \in H_n$ *such that* $A_1 = HA_2H^{-1}( \iff A_1H = HA_2)$.

PROOF. "$\Longrightarrow$": By Lemma 1.4.8, it suffices to only consider perfect deterministic strategies. We thus consider a perfect deterministic strategy for the switching isomorphism game. It is given by functions $f_A : V \to V \times \{\pm 1\}$ and $f_B : V \to V \times \{\pm 1\}$. However, since the game is synchronous Alice and Bob must give the same answer when asked the same question and therefore the two functions must be the same and we put $f \coloneqq f_A = f_B$. In order to be able to better reason about $f$, let us give names to the two parts of the function and put

$$f(v) = (\sigma(v), s(v)) \text{ for } \sigma : V \to V \text{ and } s : V \to \{\pm 1\}.$$

Considering the first two conditions of the switching isomorphism game, we notice that they coincide with the conditions for the graph isomorphism game defined in [2] and thus the function $\sigma$ must define a perfect deterministic strategy for this game. Therefore, $\sigma$ must be an isomorphism of the base graphs of $\Gamma_1$ and $\Gamma_2$ and in particular it must be a permutation of $V$. We now put $H_{ij} := \delta_{\sigma(i)j} \cdot s(i)$. Then $H$ is in $H_n$, since it has exactly one non-zero entry in each row and column and this entry is either $+1$ or $-1$. Moreover, we compute

$$\begin{aligned} HA_2H^{-1} &= \left(H_{i\sigma(i)}H_{j\sigma(j)}(A_2)_{\sigma(i)\sigma(j)}\right)_{ij} \\ &= \left(s(i)s(j)(A_2)_{\sigma(i)\sigma(j)}\right)_{ij}. \end{aligned}$$

Since the base graphs of $\Gamma_1$ and $\Gamma_2$ are isomorphic via $\sigma$, we know that

$$(A_2)_{\sigma(i)\sigma(j)} \neq 0 \iff (A_1)_{ij} \neq 0,$$

in other words we have

$$\sigma(i) \sim \sigma(j) \iff i \sim j.$$

We thus get with condition (iii) that

$$\left(s(i)s(j)(A_2)_{\sigma(i)\sigma(j)}\right)_{ij} = (A_1)_{ij}$$

and therefore

$$A_1 = HA_2H^{-1}.$$

"$\Longleftarrow$": Since $H$ is in $H_n$, we get a permutation matrix from $H$ by taking the Schur-product with itself:

$$\sigma = H \odot H.$$

Moreover, we get a sign function $s\colon V \to \{+1, -1\}$ by putting

$$s(i) = H_{i\sigma(i)}.$$

The perfect classical strategy we construct will then be given by

$$f : V \to V \times \{\pm 1\},\ f(v) := (\sigma(v), s(v))$$

for both Alice and Bob. Since Alice and Bob have the same function, it is already clear that conditions (ii) and (iv) are fulfilled. For the other conditions, consider the following: we have $A_1 = HA_2H^{-1}$ by assumption, i.e.

$$\left(H_{i\sigma(i)}H_{j\sigma(j)}(A_2)_{\sigma(i)\sigma(j)}\right)_{ij} = \left(s(i)s(j)(A_2)_{\sigma(i)\sigma(j)}\right)_{ij} = A_1.$$

Comparing the entries, we again see that

$$(A_2)_{\sigma(i)\sigma(j)} \neq 0 \iff (A_1)_{ij} \neq 0$$

and thus condition (i) will always be fulfilled. Lastly, we see for $i \sim j$ in $\Gamma_1$ that

$$\ell_1(i,j) = (A_1)_{ij} = s(i)s(j)(A_2)_{\sigma(i)\sigma(j)} = s(i)s(j)\ell_2(\sigma(i), \sigma(j))$$

and therefore condition (iii) is fulfilled. □

We now know that the switching isomorphism game captures classical switching isomorphism with its perfect classical strategies. Inspired by the definition of quantum isomorphism of graphs, we therefore define quantum switching isomorphism as follows.

5.2.3. Definition. Let $\Gamma_1$ and $\Gamma_2$ be finite, simple signed graphs. We say $\Gamma_1$ and $\Gamma_2$ are *quantum switching isomorphic*, $\Gamma_1 \cong_{q.sw.} \Gamma_2$, if there exists a perfect quantum commuting strategy for the switching isomorphism game $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$.

In the following, we will take a closer look at what it means for signed graphs to be quantum switching isomorphic. First however, we will define quantum signed permutation matrices, which have a similar relation to $H_n^+$ as quantum permutation matrices have to $S_n^+$, i.e. any matrix as defined below will yield a representation of $C(H_n^+)$.

5.2.4. Definition. An $n \times n$ matrix $v = (v_{ij})_{i,j \in \{1,\dots,n\}}$ whose entries $v_{ij}$ are elements of a unital $C^*-$algebra such that each $v_{ij}$ is a self-adjoint partial isometry and $\sum_{k=1}^n v_{ik}^2 = \sum_{k=1}^n v_{kj}^2 = 1$ for each $i, j \in \{1, \dots, n\}$ is a *quantum signed permutation matrix*. Here, an element $a$ being a partial isometry means that $aa^*a = a$. A self-adjoint partial isometry is thus an element $a$ that satisfies $a^3 = a$.

5.2.5. Lemma. *Let us assume that we have a perfect quantum commuting strategy for* $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$. *Then there exists a quantum signed permutation matrix $v$ with entries in a unital $C^*$-algebra admitting a faithful tracial state such that*

$$A_1 v = v A_2.$$

Proof. Since the game is synchronous, we can use the perfect quantum commuting strategy to apply Lemma 1.4.12 to get the unital $C^*$-algebra $\mathcal{A}$ and projections $E_{x,(y,s)} \in \mathcal{A}$ for $x \in V(\Gamma_1)$, $y \in V(\Gamma_2)$ and $s \in \{\pm 1\}$. We will write these projections as

$$E_{xy}^{(s)} := E_{x,(y,s)}.$$

By Lemma 1.4.12 we have that

$$\sum_{y,s} E_{xy}^{(s)} = 1 \text{ for all } x \in V(\Gamma_1)$$

and moreover

$$E_{x_a y_a}^{(s_a)} E_{x_b y_b}^{(s_b)} = 0$$

whenever any of the four conditions from Definition 5.2.1 is not met by $x_a, x_b, y_a, y_b, s_a$ and $s_b$. We now define the matrix $v$ via

$$v_{ij} := E_{ij}^{(+1)} - E_{ij}^{(-1)}.$$

**Step 1:** We will first show that $v$ is a quantum signed permutation matrix:

- $v_{ij}^* = E_{ij}^{(+1)*} - E_{ij}^{(-1)*} = E_{ij}^{(+1)} - E_{ij}^{(-1)} = v_{ij}$, since the $E_{ij}^{(s)}$ are projections
- $v_{ij}^2 = E_{ij}^{(+1)} E_{ij}^{(+1)} + E_{ij}^{(-1)} E_{ij}^{(-1)} = E_{ij}^{(+1)} + E_{ij}^{-1}$, since $E_{ij}^{(+1)} E_{ij}^{(-1)} = 0$, as the switching isomorphism game is synchronous. We thus get $v_{ij}^3 = E_{ij}^{(+1)} - E_{ij}^{(-1)}$.
- $v_{ik} v_{jk} = 0$ as we already know that $E_{ik}^{s_1} E_{jk}^{s_2} = 0$ for $i \neq j$, since "equality" $= \mathrm{rel}(k,k) \neq \mathrm{rel}(i,j)$.
- $\sum_{k=1}^n v_{ik}^2 = \sum_{k=1}^n E_{ik}^{(+1)} + E_{ik}^{(-1)}$ by the above, and since $\{E_{ik}^{(+1)}, E_{ik}^{(-1)}\}_k$ is a POVM for each $i$, we know that the sum must be equal to 1.
- We know that $E_{i_1 j}^{(\pm 1)} E_{i_2 j}^{(\pm 1)} = 0$ for $i_1 \neq i_2$ since Alice and Bob can not answer with the same vertices upon receiving different vertices by condition (ii). Therefore, the sum of such projections is again a projection, and in particular $\sum_{k \in V_1} \left( E_{kj}^{(+1)} + E_{kj}^{(-1)} \right)$ is a projection. Since it is a projection, we also know that
$$\sum_{\substack{k \in V_1 \\ s \in \{\pm 1\}}} E_{kj}^{(s)} \leq 1.$$
Using the above fact that $\sum_{j,s} E_{kj}^{(s)} = 1$ for any $k$, we see that
$$\sum_{k \in V_1} \sum_{\substack{j \in V_2 \\ s \in \{\pm 1\}}} E_{kj}^{(s)} = n \cdot 1.$$
Reordering this sum, we get
$$\sum_{j \in V_2} \sum_{\substack{k \in V_1 \\ s \in \{\pm 1\}}} E_{kj}^{(s)} = n \cdot 1.$$
Fixing an arbitrary $j' \in V_2$, and using $\sum_{k,s} E_{kj'}^{(s)} \leq 1$, we can reorder this to see
$$1 \geq \sum_{\substack{k \in V_1 \\ s \in \{\pm 1\}}} E_{kj'}^{(s)} = n \cdot 1 - \sum_{j' \neq j \in V_2} \underbrace{\sum_{\substack{k \in V_1 \\ s \in \{\pm 1\}}} E_{kj}^{(s)}}_{\leq 1} \geq n \cdot 1 - (n-1) \cdot 1 = 1.$$
We thus see
$$\sum_{k \in V_1} v_{kj}^2 = \sum_{k \in V_1} \left( E_{kj}^{(+1)} + E_{kj}^{(-1)} \right) = 1$$
for any $j \in V_2$.

We thus see that $v$ defined as above is indeed a quantum signed permutation matrix.

**Step 2:** Next, we will show that $A_1 v = v A_2$. Looking at the $(i,j)$-entry of the matrices, we need to show
$$(A_1 v)_{ij} = \sum_{l \sim i} \left( E_{lj}^{(+1)} - E_{lj}^{(-1)} \right) \ell_1(i,l) \stackrel{!}{=} \sum_{k \sim j} \left( E_{ik}^{(+1)} - E_{ik}^{(-1)} \right) \ell_2(k,j) = (v A_2)_{ij}.$$

For this, consider

$$\begin{aligned}\sum_{k\sim j}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\ell_2(k,j) &= \sum_{k\sim j}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\ell_2(k,j)\underbrace{\sum_{l}\left(E_{lj}^{(+1)}+E_{lj}^{(-1)}\right)}_{=1}\\ &= \sum_{k\sim j}\sum_{l}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}+E_{lj}^{(-1)}\right)\ell_2(k,j)\\ &= \sum_{k\sim j}\sum_{l\sim i}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}+E_{lj}^{(-1)}\right)\ell_2(k,j)\end{aligned}$$

.

Here, the last step is due to the fact that $E_{ik}^{s_1}E_{lj}^{s_2}=0$ if $k\sim j$ but $l\not\sim i$. If we now had that

$$\begin{aligned}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}+E_{lj}^{(-1)}\right)\ell_2(k,j)&\\ =\left(E_{ik}^{(+1)}+E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}-E_{lj}^{(-1)}\right)\ell_1(i,l),&\end{aligned}\tag{5.2.1}$$

for $k\sim j$, $i\sim l$ we would be done, since then

$$\begin{aligned}&\sum_{k\sim j}\sum_{l\sim i}\left(E_{ik}^{(+1)}-E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}+E_{lj}^{(-1)}\right)\ell_2(k,j)\\ =&\sum_{k\sim j}\sum_{l\sim i}\left(E_{ik}^{(+1)}+E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}-E_{lj}^{(-1)}\right)\ell_1(i,l)\\ =&\sum_{k}\sum_{l\sim i}\left(E_{ik}^{(+1)}+E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)}-E_{lj}^{(-1)}\right)\ell_1(i,l)\\ =&\underbrace{\sum_{k}\left(E_{ik}^{(+1)}+E_{ik}^{(-1)}\right)}_{=1}\sum_{l\sim i}\left(E_{lj}^{(+1)}-E_{lj}^{(-1)}\right)\ell_1(i,l)\\ =&\sum_{l\sim i}\left(E_{lj}^{(+1)}-E_{lj}^{(-1)}\right)\ell_1(i,l).\end{aligned}$$

Therefore, we only need to show that (5.2.1) holds. For this, we will make a case distinction on $\ell_1(i,l)$ and $\ell_2(k,j)$ for $k\sim j$, $i\sim l$.

*Case 1:* $\ell_1(i,l)=\ell_2(k,j)$. In this case, we have that

$$E_{ik}^{(+1)}E_{lj}^{(-1)}=0=E_{ik}^{(-1)}E_{lj}^{(+1)}$$

by condition (iii) of the switching isomorphism game. We compute

$$\begin{aligned}
&\left(E_{ik}^{(+1)} - E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)} + E_{lj}^{(-1)}\right)\ell_2(k,j)\\
&= \left(E_{ik}^{(+1)}E_{lj}^{(+1)} + \underbrace{E_{ik}^{(+1)}E_{lj}^{(-1)}}_{=0} - \underbrace{E_{ik}^{(-1)}E_{lj}^{(+1)}}_{=0} - E_{ik}^{(-1)}E_{lj}^{(-1)}\right)\ell_2(k,j)\\
&= \left(E_{ik}^{(+1)}E_{lj}^{(+1)} - \underbrace{E_{ik}^{(+1)}E_{lj}^{(-1)}}_{=0} + \underbrace{E_{ik}^{(-1)}E_{lj}^{(+1)}}_{=0} - E_{ik}^{(-1)}E_{lj}^{(-1)}\right)\ell_1(i,l)\\
&= \left(E_{ik}^{(+1)} + E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)} - E_{lj}^{(-1)}\right)\ell_1(i,l).
\end{aligned}$$

*Case 2:* $\ell_1(i,l) = -\ell_2(k,j)$. We then get

$$E_{ik}^{(+1)}E_{lj}^{(+1)} = 0 = E_{ik}^{(-1)}E_{lj}^{(-1)}.$$

We thus get

$$\begin{aligned}
&\left(E_{ik}^{(+1)} - E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)} + E_{lj}^{(-1)}\right)\ell_2(k,j)\\
&= \left(\underbrace{E_{ik}^{(+1)}E_{lj}^{(+1)}}_{=0} + E_{ik}^{(+1)}E_{lj}^{(-1)} - E_{ik}^{(-1)}E_{lj}^{(+1)} - \underbrace{E_{ik}^{(-1)}E_{lj}^{(-1)}}_{=0}\right)\ell_2(k,j)\\
&= \left(E_{ik}^{(+1)}E_{lj}^{(+1)} - E_{ik}^{(+1)}E_{lj}^{(-1)} + E_{ik}^{(-1)}E_{lj}^{(+1)} - E_{ik}^{(-1)}E_{lj}^{(-1)}\right)(-\ell_2(k,j))\\
&= \left(E_{ik}^{(+1)} + E_{ik}^{(-1)}\right)\left(E_{lj}^{(+1)} - E_{lj}^{(-1)}\right)\ell_1(i,l).
\end{aligned}$$

Therefore, we are done, as was argued above, and it holds indeed that

$$A_1 u = u A_2.$$

□

5.2.6. Lemma. *Let $\Gamma_1$, $\Gamma_2$ be two signed graphs and let $v$ be a quantum signed permutation matrix with entries being from a unital $C^*$-algebra $\mathcal{A}$ admitting a faithful tracial state. If we have*

$$A_1 v = v A_2,$$

*then there exists a perfect quantum commuting strategy for* SwIso$(\Gamma_1, \Gamma_2)$.

Proof. We construct projections $E_{ij}^{(s)} \in \mathcal{A}$ for $i \in V(\Gamma_1), j \in V(\Gamma_2)$ and $s \in \{\pm 1\}$ as follows:

$$E_{ij}^{(+1)} := \frac{(v_{ij}^2 + v_{ij})}{2} \qquad E_{ij}^{(-1)} := \frac{(v_{ij}^2 - v_{ij})}{2}.$$

In the following, we will show that these $E_{ij}^{(s)}$ fulfill the conditions from Lemma 1.4.12, i.e. that $\sum_{j,s} E_{ij}^{(s)} = 1$ for any $i \in V(\Gamma_1)$ and that

$$E_{ij}^{(s)} E_{kl}^{(t)} = 0$$

whenever any condition from the game $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$ is not satisfied. We then get a perfect quantum commuting strategy for the game by Lemma 1.4.12.

**Step 1:** To show that condition (iv) is fulfilled, we first note that the $E_{ij}^{\pm 1}$ are indeed projections, since the $v_{ij}$ are selfadjoint partial isometries:

$$\left(E_{ij}^{\pm 1}\right)^2 = \frac{(v_{ij}^2 \pm v_{ij})(v_{ij}^2 \pm v_{ij})}{4} = \frac{v_{ij}^4 \pm 2v_{ij}^3 + v_{ij}^2}{4} = \frac{v_{ij}^2 \pm 2v_{ij} + v_{ij}^2}{4} = \frac{v_{ij}^2 \pm v_{ij}}{2}$$
$$\left(E_{ij}^{\pm 1}\right)^* = \frac{(v_{ij}^2)^* \pm v_{ij}^*}{2} = \frac{v_{ij}^2 \pm v_{ij}}{2}.$$

This yields that $E_{ij}^{(+1)}$ and $E_{ij}^{(-1)}$ are orthogonal:

$$\frac{(v_{ij}^2 + v_{ij})(v_{ij}^2 - v_{ij})}{4} = \frac{v_{ij}^4 - v_{ij}^2}{4} = \frac{v_{ij}^2 - v_{ij}^2}{4} = 0.$$

We thus get that condition (iv) of the switching isomorphism game is satisfied by the operators. Also, note that we have

$$E_{ij}^{(+1)} - E_{ij}^{(-1)} = \frac{\left(v_{ij}^2 + v_{ij}\right)}{2} - \frac{\left(v_{ij}^2 - v_{ij}\right)}{2} = \frac{v_{ij}^2 + v_{ij} - v_{ij}^2 + v_{ij}}{2} = v_{ij}$$

and thus, since $E_{ij}^{(+1)}$ and $E_{ij}^{(-1)}$ are orthogonal,

$$v_{ij}^2 = E_{ij}^{(+1)} + E_{ij}^{(-1)}.$$

**Step 2:** We will next show, that conditions (i) and (ii) are satisfied by the constructed projections. For this, we assume that $\mathrm{rel}(i,k) \neq \mathrm{rel}(j,l)$ and show that then

$$E_{ij}^{(\pm 1)} E_{kl}^{(\pm 1)} = 0.$$

This proof is mostly the same as the corresponding proof for the isomorphism game in [2]. We will make it as a case distinction on $\mathrm{rel}(i,k)$ and $\mathrm{rel}(j,l)$:

*Case 1:* $i = k, j \neq l$: Since $v$ is a quantum signed permutation matrix, we have

$$\sum_j v_{ij}^2 = 1.$$

But as we established above, we have $v_{ij}^2 = E_{ij}^{(+1)} + E_{ij}^{(-1)}$ and thus

$$\sum_j \left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right) = 1.$$

We thus have

$$E_{ij}^{(\pm 1)} E_{il}^{(\pm 1)} = 0$$

since they are projections summing up to 1. In the same way, we see that for $i \neq k$ but $j = l$, we have

$$E_{ij}^{(\pm 1)} E_{kj}^{(\pm 1)} = 0.$$

*Case 2:* $i \sim k, j \not\sim l$: In this case, if we had $j = l$, we would be in the above case. Let therefore $j \neq l$. Let us now take a look at the $(i,l)$-entries of $A_1 v$ and

$vA_2$:

$$(A_1v)_{il} = \sum_{g\sim i} v_{gl}\ell_1(i,g) = \sum_{g\sim i}\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g)$$

$$(vA_2)_{il} = \sum_{h\sim l} v_{ih}\ell_2(h,l) = \sum_{h\sim l}\left(E_{ih}^{(+1)} - E_{ih}^{(-1)}\right)\ell_2(h,l).$$

Since by assumption $A_1v = vA_2$, we know that

$$\sum_{g\sim i}\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g) = \sum_{h\sim l}\left(E_{ih}^{(+1)} - E_{ih}^{(-1)}\right)\ell_2(h,l).$$

From this, we get

$$\sum_{h'\sim l}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\sum_{g\sim i}\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g)$$

$$=\sum_{h'\sim l}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\sum_{h\sim l}\left(E_{ih}^{(+1)} - E_{ih}^{(-1)}\right)\ell_2(h,l)$$

and since $E_{ih'}^{(\pm1)}E_{ih}^{(\pm1)} = 0$ for $h' \neq h$

$$=\sum_{h'}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\sum_{h\sim l}\left(E_{ih}^{(+1)} - E_{ih}^{(-1)}\right)\ell_2(h,l)$$

$$=\sum_{h'}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\sum_{g\sim i}\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g)$$

$$\implies \sum_{h'\not\sim l}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\sum_{g\sim i}\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g) = 0.$$

We can now rearrange the above equation to get

$$\left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right)\left(E_{kl}^{(+1)} - E_{kl}^{(-1)}\right)\ell_1(i,k) = \sum_{\substack{h'\not\sim l\\ h'\neq j}}\sum_{\substack{g\sim i\\ g\neq k}}\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)\left(E_{gl}^{(+1)} - E_{gl}^{(-1)}\right)\ell_1(i,g).$$

Multiplying both sides with $\left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right)$ from the left yields

$$\underbrace{\left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right)^2}_{=\left(E_{ij}^{(+1)}+E_{ij}^{(-1)}\right)}\left(E_{kl}^{(+1)} - E_{kl}^{(-1)}\right)\ell_1(i,k) = 0 \tag{5.2.2}$$

since on the right hand side, we have terms of the form

$$\left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right)\left(E_{ih'}^{(+1)} + E_{ih'}^{(-1)}\right)$$

appearing in every summand, but these are orthogonal for $h' \neq j$, as was shown in step 5.2. Multiplying with $\ell_1(i,k)$ on both sides and expanding in (5.2.2), we get

$$E_{ij}^{(+1)}E_{kl}^{(+1)} + E_{ij}^{(-1)}E_{kl}^{(+1)} - E_{ij}^{(+1)}E_{kl}^{(-1)} - E_{ij}^{(-1)}E_{kl}^{(-1)} = 0. \tag{5.2.3}$$

Since we know, that for any vertices $g, h$ and any signs $a \neq b$ we have

$$E_{gh}^{(a)} E_{gh}^{(b)} = 0,$$

we can isolate any single summand $E_{ij}^{(c)} E_{kl}^{(d)}$ in (5.2.3), by multiplying both sides from the left with $E_{ij}^{(c)}$ and both sides from the right with $E_{kl}^{(d)}$. We thus get

$$E_{ij}^{(\pm 1)} E_{kl}^{(\pm 1)} = 0,$$

as desired.

We now have shown that conditions (i) and (ii) hold if one relation is "equality" and the other is not and if one relation is adjacency and the other is not. What is still missing is the case where the relation of $i$ and $k$ is $i \not\sim k$, $i \neq k$, and the relation of $j$ and $l$ is different from this. However, then we either have $j = l$ and are in the first case, or we have $j \sim l$ and are in the second case. Therefore, conditions (i) and (ii) hold.

**Step 3:** It remains to be shown, that condition (iii) is satisfied. For this, let $i \sim k$ and $j \sim l$ be vertices. We want to show that $E_{ij}^{(s_1)} E_{kl}^{(s_2)} = 0$ for all $s_1$, $s_2$ for which we have $\ell_1(i,k) \neq s_1 s_2 \ell_2(l,j)$.

We look at the $(k,j)$-entries of $A_1 v$ and of $v A_2$ and know by assumption

$$(A_1 v)_{kj} = \sum_{g \sim k} \left(E_{gj}^{(+1)} - E_{gj}^{(-1)}\right) \ell_1(k,g) = \sum_{h \sim j} \left(E_{kh}^{(+1)} - E_{kh}^{(-1)}\right) \ell_2(h,j) = (v A_2)_{kj}\,.$$

Multiplying these two entries together, we get

$$\begin{aligned}\sum_{\substack{g \sim k \\ h \sim j}} \left(E_{gj}^{(+1)} - E_{gj}^{(-1)}\right)\left(E_{kh}^{(+1)} - E_{kh}^{(-1)}\right) \ell_1(k,g)\ell_2(h,j) &= \sum_{g \sim k} \left(E_{gj}^{(+1)} - E_{gj}^{(-1)}\right)^2 \\ &= \sum_{g \sim k} \left(E_{gj}^{(+1)} + E_{gj}^{(-1)}\right)\end{aligned}$$

but also

$$\begin{aligned}&\sum_{\substack{g \sim k \\ h \sim j}} \left(E_{gj}^{(+1)} - E_{gj}^{(-1)}\right)\left(E_{kh}^{(+1)} - E_{kh}^{(-1)}\right) \ell_1(k,g)\ell_2(h,j) \\ =&\sum_{\substack{g \sim k \\ h \sim j}} \left(E_{gj}^{(+1)} E_{kh}^{(+1)} - E_{gj}^{(+1)} E_{kh}^{(-1)} - E_{gj}^{(-1)} E_{kh}^{(+1)} + E_{gj}^{(-1)} E_{kh}^{(-1)}\right) \ell_1(k,g)\ell_2(h,j).\end{aligned}$$

Putting this together, we get

$$\begin{aligned}\sum_{\substack{g \sim k \\ h \sim j}} \left(E_{gj}^{(+1)} E_{kh}^{(+1)} - E_{gj}^{(+1)} E_{kh}^{(-1)} - E_{gj}^{(-1)} E_{kh}^{(+1)} + E_{gj}^{(-1)} E_{kh}^{(-1)}\right) \ell_1(k,g)\ell_2(h,j) \\ = \sum_{g \sim k} \left(E_{gj}^{(+1)} + E_{gj}^{(-1)}\right).\end{aligned} \tag{5.2.4}$$

To use this, we will do a case distinction on $\ell_1(i,k)$ and $\ell_2(l,j)$:

*Case 1:* $\ell_1(i,k) = \ell_2(l,j)$

In this case, we have that $\ell_1(i,k)\ell_2(l,j) = +1$. This means that the term in the sum on the left hand side in (5.2.4) for $g = i$, $h = l$ is of the form

$$E_{ij}^{(+1)}E_{kl}^{(+1)} - E_{ij}^{(+1)}E_{kl}^{(-1)} - E_{ij}^{(-1)}E_{kl}^{(+1)} + E_{ij}^{(-1)}E_{kl}^{(-1)}.$$

Since we are interested in $E_{ij}^{(+1)}E_{kl}^{-1}$ and $E_{ij}^{(-1)}E_{kl}^{(+1)}$, we can now just isolate them with positive sign on one side of the equation. Rearranging the terms in (5.2.4) in this manner, we get

$$\begin{aligned}E_{ij}^{(+1)}E_{kl}^{(-1)} + E_{ij}^{(-1)}E_{kl}^{(+1)} = \\ \sum_{\substack{g\sim k\\ g\neq i}}\sum_{\substack{h\sim j\\ h\neq l}}\left(E_{gj}^{(+1)}E_{kh}^{(+1)} - E_{gj}^{(+1)}E_{kh}^{(-1)} - E_{gj}^{(-1)}E_{kh}^{(+1)} + E_{gj}^{(-1)}E_{kh}^{(-1)}\right)\ell_1(k,g)\ell_2(h,j) \\ + E_{ij}^{(+1)}E_{kl}^{(+1)} + E_{ij}^{(-1)}E_{kl}^{(-1)} - \sum_{g\sim k}\left(E_{gj}^{(+1)} + E_{gj}^{(-1)}\right).\end{aligned}$$

By multiplying the above equation from the left with $E_{ij}^{(+1)}$ and from the right with $E_{kl}^{(-1)}$, we see that most terms become 0:

- the big sum has a term $E_{gj}^{(\pm1)}$ to the very left of each summand, where $g \neq i$, and thus we get $E_{ij}^{(+1)}E_{gj}^{(\pm1)} = 0$ in each summand and thus the big sum is 0
- $\left(E_{ij}^{(+1)}\right)^2 E_{kl}^{(+1)}E_{kl}^{(-1)} = 0$, since $E_{kl}^{(+1)}E_{kl}^{(-1)} = 0$
- $E_{ij}^{(+1)}E_{ij}^{(-1)}\left(E_{kl}^{(-1)}\right)^2 = 0$, since $E_{ij}^{(+1)}E_{ij}^{(-1)} = 0$
- in the last sum, all summands for $g \neq i$ are 0, and we are left with

$$E_{ij}^{(+1)}\left(E_{ij}^{(+1)} + E_{ij}^{(-1)}\right)E_{kl}^{(-1)} = E_{ij}^{(+1)}E_{kl}^{(-1)}$$

  since $E_{ij}^{(+1)}E_{ij}^{(-1)} = 0$.

On the left hand side, only the term $E_{ij}^{(+1)}E_{kl}^{(-1)}$ remains and putting it together, we get

$$E_{ij}^{(+1)}E_{kl}^{(-1)} = -E_{ij}^{(+1)}E_{kl}^{(-1)}.$$

But this is only possible, if

$$E_{ij}^{(+1)}E_{kl}^{(-1)} = 0.$$

In the same manner, but multiplying with $E_{ij}^{(-1)}$ from the left and $E_{kl}^{(+1)}$ from the right, we get

$$E_{ij}^{(-1)}E_{kl}^{(+1)} = -E_{ij}^{(-1)}E_{kl}^{(+1)} \implies E_{ij}^{(-1)}E_{kl}^{(+1)} = 0.$$

We thus showed that $\ell_1(i,k) = \ell_2(l,j)$ implies

$$E_{ij}^{(+1)}E_{kl}^{(-1)} = 0 = E_{ij}^{(-1)}E_{kl}^{(+1)},$$

as desired.

*Case 2:* $\ell_1(i,k) \neq \ell_2(l,j)$

In this case, we have $\ell_1(i,k)\ell_2(l,j) = -1$ and the term in the left hand sum in (5.2.4) for $g = i$, $h = l$ is of the form

$$-E_{ij}^{(+1)}E_{kl}^{(+1)} + E_{ij}^{(+1)}E_{kl}^{(-1)} + E_{ij}^{(-1)}E_{kl}^{(+1)} - E_{ij}^{(-1)}E_{kl}^{(-1)}.$$

Using the same reasoning as in case 1, we isolate the terms we are interested in, namely $E_{ij}^{(+1)}E_{kl}^{(+1)}$ and $E_{ij}^{(-1)}E_{kl}^{(-1)}$, with positive sign on one side and collect the rest on the other side. We get

$$\begin{aligned} &E_{ij}^{(+1)}E_{kl}^{(+1)} + E_{ij}^{(-1)}E_{kl}^{(-1)} = \\ &\sum_{\substack{g\sim k\\ g\neq i}}\sum_{\substack{h\sim j\\ h\neq l}} \left(E_{gj}^{(+1)}E_{kh}^{(+1)} - E_{gj}^{(+1)}E_{kh}^{(-1)} - E_{gj}^{(-1)}E_{kh}^{(+1)} + E_{gj}^{(-1)}E_{kh}^{(-1)}\right)\ell_1(k,g)\ell_2(h,j) \\ &\qquad\qquad + E_{ij}^{(+1)}E_{kl}^{(-1)} + E_{ij}^{(-1)}E_{kl}^{(+1)} - \sum_{g\sim k}\left(E_{gj}^{(+1)} + E_{gj}^{(-1)}\right). \end{aligned}$$

Now, we multiply with $E_{ij}^{(+1)}$ from the left and $E_{kl}^{(+1)}$ from the right to get

$$E_{ij}^{(+1)}E_{kl}^{(+1)} = -E_{ij}^{(+1)}E_{kl}^{(+1)} \implies E_{ij}^{(+1)}E_{kl}^{(+1)} = 0$$

and multiply with $E_{ij}^{(-1)}$ from the left and $E_{kl}^{(-1)}$ from the right to get

$$E_{ij}^{(-1)}E_{kl}^{(-1)} = -E_{ij}^{(-1)}E_{kl}^{(-1)} \implies E_{ij}^{(-1)}E_{kl}^{(-1)} = 0$$

We thus have

$$\ell_1(i,k) \neq \ell_2(l,j) \implies E_{ij}^{(+1)}E_{kl}^{(+1)} = 0 = E_{ij}^{(-1)}E_{kl}^{(-1)},$$

as desired.

Therefore, also condition (iii) of the switching isomorphism game holds and thus by Lemma 1.4.12 there exists a perfect quantum commuting strategy for $\mathrm{SwIso}(\Gamma_1, \Gamma_2)$. □

Together, the two lemmas above yield the following theorem.

5.2.7. Theorem. *Let $\Gamma_1$ and $\Gamma_2$ be finite, simple signed graphs with adjacency matrices $A_1$ and $A_2$. $\Gamma_1$ and $\Gamma_2$ are quantum switching isomorphic if and only if there exists a quantum signed permutation matrix $v$ with entries from a unital $C^*$-algebra admitting a faithful tracial state such that $vA_1 = A_2v$.*

Proof. This follows immediately from Lemma 5.2.5 and Lemma 5.2.6. □

## 5.3. Link of Quantum Switching Isomorphism to Quantum Isomorphism

The first question one can ask when defining a generalised kind of symmetry as above is whether this does in fact give any "new" symmetries. In the case of quantum isomorphisms of graphs it was shown in [2] that there exists a pair of graphs that are not classically isomorphic but are quantum isomorphic. Using this result it is trivial to get the same result for quantum switching isomorphisms: by taking the same pair of non-isomorphic but quantum isomorphic graphs and adding the label $+1$ to all edges, we get two graphs that are not switching isomorphic (since the base graphs are not isomorphic) but quantum switching isomorphic, using the same quantum strategy as in [2] with the additional response of $s_A = s_B = 1$ for all inputs.

The question we really want to ask is therefore: are there any signed graphs that are not switching isomorphic but quantum switching isomorphic, with the quantum switching isomorphism not coming purely from a quantum isomorphism of the base graphs? For example, are there any graphs that are not switching isomorphic, but quantum switching isomorphic, while the delabelled versions of the graphs are isomorphic?

In order to answer the question, we first present a lemma that represents a quantification of the characterisation of switching isomorphism from Definition 5.1.2.

5.3.1. Lemma. *Let $\Gamma_1$ and $\Gamma_2$ be two finite, simple signed graphs on $n$ vertices. Then $\Gamma_1 \cong_{q.sw.} \Gamma_2$ if and only if there exists a quantum permutation matrix $u$ and self-adjoint unitaries $s_i$ for each $1 \leq i \leq n$ such that*

$$(A_1)_{ij}\mathbf{1} = s_i(uA_2u^*)_{ij}s_j \quad \text{for all } 1 \leq i, j \leq n$$

*and such that $u_{ij}$ and $s_i$ commute for all $i, j$. Moreover, the quantum permutation matrix $u$ implements a quantum isomorphism of the delabelled versions of $\Gamma_1$ and $\Gamma_2$, i.e. if $\tilde{\mathrm{A}}_1$ and $\tilde{\mathrm{A}}_2$ are the adjacency matrices of the delabelled graphs, we have*

$$\tilde{\mathrm{A}}_1\, u = u\, \tilde{\mathrm{A}}_2\, .$$

*In particular, if $v$ is a quantum signed permutation matrix implementing the quantum switching isomorphism, then we have $u_{ij} = v_{ij}^2$ and $s_i = \sum_k v_{ik}$. On the other hand, if $u$ and $s_i$ are the quantum permutation matrix and the self-adjoint unitaries as above, then $v_{ij} = s_i u_{ij}$.*

Proof. We first note that given a quantum signed permutation matrix $v$, squaring the entries gives a quantum permutation matrix, since by definition $u_{ij} = v_{ij}^2$ is a projection and the sums over the rows and columns are 1. Moreover we have

$$s_i u_{ij} = \sum_k v_{ik} v_{ij}^2 = v_{ij}^3 = v_{ij},$$

since $v_{ij}$ and $v_{ik}$ are orthogonal for $j \neq k$. Similarly, we get $u_{ij}s_i = v_{ij}$ and thus they also commute. Lastly, given a quantum permutation matrix $u$ and the self-adjoint unitaries $s_i$ as above, then

$$(s_i u_{ij})^2 = s_i^2 u_{ij} = 1u_{ij} = u_{ij}$$

and

$$(s_i u_{ij})^3 = s_i u_{ij}.$$

i.e. $s_i u_{ij}$ is a self-adjoint partial isometry and thus the matrix $(s_i u_{ij})_{ij}$ is a quantum signed permutation matrix.

We now compute

$$\begin{aligned}(vA_2v^*)_{ij} &= \sum_{k,l} v_{ik}(A_2)_{kl}v_{jl} = \sum_{k,l} s_i u_{ik}(A_2)_{kl} s_j u_{jl} \\ &= s_i \sum_{k,l} u_{ik}(A_2)_{kl} u_{jl}\, s_j = s_i(uA_2u^*)_{ij} s_j.\end{aligned}$$

Since we have

$$A_1 = vA_2v^*$$

by assumption, this proves the first part of the statement.

Next, we note that if $v$ is a quantum switching isomorphism of $\Gamma_1$ and $\Gamma_2$ then the entries $v_{ij}$ encode a perfect quantum strategy as shown in Lemma 5.2.6. In that proof, we note that the elements $E_{ij}^{(+1)}$ and $E_{ij}^{(-1)}$ each satisfy that $E_{ij}^{(\pm 1)} E_{kl}^{(\pm 1)} = 0$ whenever condition (i) and (ii) in the quantum switching isomorphism game are not met. In particular, also their sum, i.e. $E_{ij}^{(+1)} + E_{ij}^{(-1)} = u_{ij}$ satisfies this, and conditions (i) and (ii) are the same as the conditions from the isomorphism game of the delabelled versions of the graphs. By Theorem 2.2.1 from [72] the matrix $u$ therefore satisfies

$$\tilde{\mathrm{A}}_1\, u = u\, \tilde{\mathrm{A}}_2\, .$$

□

Using Lemma 5.3.1, we can answer the question whether quantum switching isomorphism yields any new symmetries for connected graphs.

5.3.2. Theorem. *Let $\Gamma_1$ and $\Gamma_2$ be two connected signed graphs that are quantum switching isomorphic. If $v$ is a quantum signed permutation matrix such that $A_1 v = vA_2$ and the corresponding quantum permutation matrix $u$ as in Lemma 5.3.1 has commuting entries, then also all entries $v_{ij}$ of $v$ commute with each other.*

*Equivalently, if $v$ has any noncommuting entries, then already $u$ must have noncommuting entries.*

Proof. We will show inductively for vertices $i$ and $j$ of $\Gamma_1$ that if they are connected by a path of length $N$, consisting of the vertices $i \sim r_1 \sim \ldots \sim r_{N-1} \sim j$, then for any vertices $k, l$ of $\Gamma_2$ there exist numbers $a_{s_{N-1}} \in \{\pm 1\}$ such that

$$v_{ik}v_{jl} = \sum_{\substack{s_1,\ldots,s_{N-1} \\ k\sim s_1\sim\ldots\sim s_{N-1}\sim l}} a_{s_{N-1}} u_{ik} u_{r_1 s_1} \ldots u_{r_{N-1}s_{N-1}} u_{jl}$$

and that

$$v_{jl}v_{ik} = \sum_{\substack{s_1,\dots,s_{N-1}\\ k\sim s_1\sim\dots\sim s_{N-1}\sim l}} a_{s_{N-1}} u_{jl}u_{r_{N-1}s_{N-1}}\dots u_{r_1s_1}u_{ik}.$$

In particular, we show that $v_{ik}v_{jl} = v_{jl}v_{ik}$, since all entries of $u$ commute with each other by assumption.

For $N = 1$ we have that $i \sim j$ and use the facts from Lemma 5.3.1 that $(A_1)_{ij}\mathbf{1} = s_i(uA_2u^*)_{ij}s_j$ and that $\mathbf{1} = (\tilde{A}_1)_{ij}\mathbf{1} = (u\tilde{A}_2u^*)_{ij}$ to get

$$s_i\left(\sum_{k,l}(A_2)_{kl}u_{ik}u_{jl}\right)s_j = (A_1)_{ij}\mathbf{1} = (A_1)_{ij}(\tilde{A}_1)_{ij}\mathbf{1} = (A_1)_{ij}\sum_{\substack{k,l\\ k\sim l}} u_{ik}u_{jl}.$$

For any vertices $k' \sim l'$ of $\Gamma_2$ we can now multiply the above equation with $u_{ik'}$ from the left and $u_{jl'}$ from the right to get

$$\underbrace{v_{ik'}u_{ik'}}_{=v_{ik'}}\,\underbrace{u_{jl'}v_{jl'}}_{=v_{jl'}} = (A_1)_{ij}u_{ik'}u_{jl'},$$

where we use that $s_au_{ab} = v_{ab}$ for any vertices $a, b$ and that $v_{ab}u_{ac} = 0$ for $b \neq c$.

With a similar computation we get that

$$v_{jl'}v_{ik'} = (A_1)_{ji}u_{jl'}u_{ik'}$$

and since $(A_1)_{ij} = (A_1)_{ji}$ we get the claim with $a = (A_1)_{ij}$.

For any vertices $k' \not\sim l'$ of $\Gamma_2$, one can see that $v_{ik'}v_{jl'} = 0 = u_{ik'}u_{jl'}$ and also $v_{jl'}v_{ik'} = 0 = u_{jl'}u_{ik'}$ by Lemma 5.2.6.

Let us now assume the claim holds for any path with length up to $N$ and show it for $N+1$. By assumption, we have a path $i \sim r_1 \sim \dots \sim r_N \sim j$. Let now two vertices $k$ and $l$ of $\Gamma_2$ be given and consider the following product:

$$v_{ik}v_{jl} = v_{ik}\sum_{s_N} u_{r_Ns_N}v_{jl} = v_{ik}\sum_{s_N\sim l} u_{r_Ns_N}v_{jl} = v_{ik}\sum_{s_N\sim l} v_{r_Ns_N}^2 v_{jl}.$$

We can now apply the induction hypothesis to $v_{ik}$ and all the $v_{r_Ns_N}$ and also to $v_{r_Ns_N}$ and $v_{jl}$ to get

$$v_{ik}v_{r_Ns_N} = \sum_{\substack{s_1,\dots,s_{N-1}\\ k\sim s_1\sim\dots\sim s_{N-1}\sim s_N}} a^{(1)}_{s_{N-1}} u_{ik}u_{r_1s_1}\dots u_{r_{N-1}s_{N-1}}u_{r_Ns_N}$$

$$v_{r_Ns_N}v_{jl} = a^{(2)}_{s_N}u_{r_Ns_N}u_{jl}.$$

Combining this, we get

$$v_{ik}v_{jl} = \sum_{\substack{s_1,\dots,s_N\\ k\sim s_1\sim\dots\sim s_N\sim l}} a^{(1)}_{s_{N-1}}a^{(2)}_{s_N} u_{ik}u_{r_1s_1}\dots \underbrace{u_{r_Ns_N}u_{r_Ns_N}}_{=u_{r_Ns_N}} u_{jl}.$$

Doing the same computation for $v_{jl}v_{ik}$ we get

$$v_{jl}v_{ik} = \sum_{\substack{s_1,\dots,s_N\\ k\sim s_1\sim\dots\sim s_N\sim l}} a^{(1)}_{s_{N-1}}a^{(2)}_{s_N} u_{jl}u_{r_Ns_N}\dots u_{r_1s_1}u_{ik}$$

and all in all we get the desired outcome. □

5.3.3. Corollary. *From the above theorem it follows immediately that if two connected signed graphs* $\Gamma_1$ *and* $\Gamma_2$ *are quantum switching isomorphic, then their delabelled versions* $\tilde{\Gamma}_1$ *and* $\tilde{\Gamma}_2$ *are already quantum isomorphic.*

*In other words, the quantum isomorphism* $\tilde{\Gamma}_1 \cong_q \tilde{\Gamma}_2$ *is a prerequisite for the quantum switching isomorphism* $\Gamma_1 \cong_{q.sw.} \Gamma_2$.

CHAPTER 6

# Quantum Automorphism Groups of Matroids

This section is based on the article [26] of which the author is a co-author. Inspired by the study of quantum automorphism groups of graphs, we want to extend the concept of quantum symmetries to other combinatorial objects. In this section we thus propose several definitions for quantum automorphism groups of matroids. Introduced by Whitney [88] in the 1930's, a matroid is a common combinatorial generalisation of linear dependence of vectors and cycles in a graph. Matroids are notorious for their cryptomorphic definitions; we focus on the characterisations involving independent sets, bases, flats, and circuits. Each axiom system determines the same classical automorphism group of a matroid $\mathsf{M}$. We define quantum automorphism groups of the matroid $\mathsf{M}$

$$G_{aut}^{\mathcal{I}}(\mathsf{M}),\ G_{aut}^{\mathcal{B}}(\mathsf{M}),\ G_{aut}^{\mathcal{F}}(\mathsf{M}),\ G_{aut}^{\mathcal{C}}(\mathsf{M})$$

by adding relations to those of $S_n^+$ derived from the independent sets, bases, flats, and circuits definitions of $\mathsf{M}$, respectively, in the spirit of Bichon and Banica. We have the surprising feature that each of these yields an, a priori, different definition of a quantum automorphism group. However, for nice classes of matroids, we have a chain of inclusions, similar to the case of Bichon's and Banica's definitions of quantum automorphism groups of graphs. Our main results are summarized in the following theorem; see Section 6.2 for details on the terminology.

Throughout this chapter, we will usually denote the classical and quantum symmetric groups by $S_E$ and $S_E^+$ respectively for a given finite set $E$ of cardinality $n$ instead of writing them as $S_n$ and $S_n^+$ in order to give emphasis to the set on which they act. This is unusual in the context of quantum groups, however in the context of automorphism groups of matroids it is useful since one often wants to make statements about the concrete set $E$ and not only the number of elements of $E$.

6.1. Theorem. *For every matroid* $\mathsf{M}$ *we have*

$$C(\mathrm{Aut}(\mathsf{M})) = G_{aut}^{\mathcal{F}}(\mathsf{M}) \subseteq G_{aut}^{\mathcal{B}}(\mathsf{M}) = G_{aut}^{\mathcal{I}}(\mathsf{M}).$$

*If* $\mathsf{M}$ *is a simple rank* $3$ *matroid and the ground set* $E(\mathsf{M})$ *is not equal to* $F_1 \cup F_2 \cup F_3$ *for triangles* $\{F_1, F_2, F_3\}$*, then*

$$C(\mathrm{Aut}(\mathsf{M})) = G_{aut}^{\mathcal{F}}(\mathsf{M}) \subseteq G_{aut}^{\mathcal{C}}(\mathsf{M}) \subseteq G_{aut}^{\mathcal{B}}(\mathsf{M}) = G_{aut}^{\mathcal{I}}(\mathsf{M}).$$

We should point out that here we follow a decidedly algebraic approach, ignoring topological ramifications of Banach algebras. That is, our quantum permutation groups are defined as certain noncommutative algebras, equipped with an involution and a coproduct. In this way we obtain computational methods for constructing interesting examples. Using the free open-source computer algebra system `OSCAR` [28], [66], which is written in `Julia`, we develop code to compute $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$, and determine whether these are commutative. In particular, we find several examples where $G^{\mathcal{B}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{C}}_{aut}(\mathsf{M})$ and others where $G^{\mathcal{C}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{B}}_{aut}(\mathsf{M})$.

In Section 6.6, we derive a matroidal analog of Lovász's graph homomorphism count theorem. This theorem asserts that two graphs are isomorphic if and only if the number of graph homomorphisms from $\Gamma$ *to* these graphs are equal for all graphs $\Gamma$. By the main result of [55], two graphs are quantum isomorphic, in the sense of Banica, if these homomorphism counts agree for all *planar* graphs. We take the first steps in investigating the connection between (quantum) isomorphic matroids and counts of matroid maps (i.e., *strong maps*) by proving an analog of Lovász's theorem for matroids. The crucial difference is that matroid isomorphism is determined by strong map counts *from* the candidate matroids.

Our work is just the beginning of the study of quantum symmetry for matroids. Our hope is to provide insight into the world of matroids by dividing the class of matroids into subclasses of "more complicated/richer" matroids (having a high degree of quantum symmetry) and "easier" ones (with a low or no degree of quantum symmetry).

**Code.** The code used to collect the data recorded in Section 6.5 may be found at

`https://github.com/dmg-lab/QuantumAutomorphismGroups.jl`.

This `GitHub` repository also contains noncommutative Gröbner bases for the quantum automorphism groups obtained, stored in the `mrdi` file format [29].

## 6.1. Matroids and their Automorphism Groups

Matroids admit numerous cryptomorphic axiomatic systems. We recommend to the reader the textbook [67] as a general reference for matroid theory. We will introduce these different systems shortly in the following, with the definitions and statements coming from the mentioned textbook [67].

We begin by giving the definition via independent sets.

6.1.1. DEFINITION. A *matroid* $\mathsf{M} = (E, \mathcal{I})$ consists of a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ that satisfies the following conditions:

(I1) $\varnothing \in \mathcal{I}$.
(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
(I3) If $I_1$ and $I_2$ are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e$ of $I_2 \setminus I_1$, such that $I_1 \cup \{e\} \in \mathcal{I}$.

Here, condition (I3) is called the *independence augmentation axiom.*

If $\mathsf{M} = (E, \mathcal{I})$ is a matroid, then we call $\mathsf{M}$ the *matroid on* $E$. We moreover call the members of $\mathcal{I}$ the *independent sets* of $\mathsf{M}$ and $E$ the *ground set* of $\mathsf{M}$. A subset of $E$ that is not in $\mathcal{I}$ is called *dependent.* A *minimal dependent set*, i.e. a dependent set all of whose proper subsets are independent, is called a *circuit* of $\mathsf{M}$ and the set of all circuits will be denoted by $\mathcal{C}$. If we want to emphasize the dependence on the matroid, we shall sometimes write $E(\mathsf{M})$, $\mathcal{I}(\mathsf{M})$ and $\mathcal{C}(\mathsf{M})$.

An alternative way to characterise a matroid is via the set of circuits.

6.1.2. THEOREM. *Let $E$ be a finite ground set and let $\mathcal{C}$ be a collection of subsets of $E$. Let $\mathcal{I}$ be the collection of subsets of $E$ that contain no member of $\mathcal{C}$. The following are equivalent.*

- $\mathsf{M} = (E, \mathcal{I})$ *is a matroid with set of circuits given by* $\mathcal{C}$.
- $\mathcal{C}$ *satisfies*
  *(C1)* $\varnothing \notin \mathcal{C}$.
  *(C2) If $C_1$ and $C_2$ are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*
  *(C3) If $C_1$ and $C_2$ are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member $C_3$ of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*

Condition (C3) is called the *circuit elimination axiom.*

Instead of looking at minimal dependent sets, one can also look at maximal independent sets as an efficient way to specify all independent sets of a matroid. We call these maximal independent sets *bases* of the matroid and write $\mathcal{B}$ or $\mathcal{B}(\mathsf{M})$ for the set of all bases of a matroid.

Using bases, one can characterise a matroid as follows:

6.1.3. THEOREM. *Let $E$ be a finite ground set and let $\mathcal{B}$ be a collection of subsets of $E$. Let $\mathcal{I}$ be the collection of subsets of $E$ that are contained in some member of $\mathcal{B}$. The following are equivalent:*

- $\mathsf{M} = (E, \mathcal{I})$ *is a matroid with collection of bases* $\mathcal{B}$.
- $\mathcal{B}$ *satisfies*
  *(B1)* $\mathcal{B}$ *is non-empty.*
  *(B2) If $B_1$ and $B_2$ are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element $y$ of $B_2 \setminus B_1$, such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

The axiom (B2) is one of several *basis exchange axioms* that hold for matroids. A basic fact about bases of matroids is, unsurprisingly, that all bases of a matroid have the same cardinality.

6.1.4. LEMMA. *Let $B_1$ and $B_2$ be bases of a matroid $\mathsf{M}$. Then $|B_1| = |B_2|$.*

Another notion from linear algebra that can be generalised to the matroid setting is that of the dimension of a vector space. In the context of matroids, this is done via a *rank function.*

6.1.5. DEFINITION. Let $\mathsf{M} = (E, \mathcal{I})$ be a matroid and let $X \subseteq E$ be a subset of the ground set. Let $\mathcal{I}|X := \{I \subseteq X : I \in \mathcal{I}\}$ be the restriction of $\mathcal{I}$ to $X$. Then it is easy to see that $(X, \mathcal{I}|X)$ is also a matroid, which is called the *restriction of* $\mathsf{M}$ *to* $X$ or the *deletion of* $E \setminus X$ *from* $\mathsf{M}$. We also write $\mathsf{M}|X$ or $\mathsf{M} \setminus (E \setminus X)$ for this matroid.

Since by Lemma 6.1.4 all bases of a matroid have the same cardinality, and since $\mathsf{M}|X$ is a matroid, we define the *rank* $\mathrm{rank}(X)$ of $X$ to be the size of a basis $B$ of $\mathsf{M}|X$ and call such a set $B$ a *basis* of $X$.

We call this function rank, or sometimes also $\mathrm{rank}_{\mathsf{M}}$, the *rank function* of $\mathsf{M}$, which maps $2^E$ to the set of non-negative integers. We will also sometimes write $\mathrm{rank}(\mathsf{M})$ to mean $\mathrm{rank}(E(\mathsf{M}))$.

Once again, one can give an alternative characterisation of matroids using the rank function.

6.1.6. THEOREM. *Let $E$ be a finite ground set. Let* rank *be a function from $2^E$ to the non-negative integers and let $\mathcal{I}$ be the collection of subsets $X$ of $E$ for which* $\mathrm{rank}(X) = |X|$. *The following are equivalent:*

- $\mathsf{M} = (E, \mathcal{I})$ *is a matroid with rank function* rank.
- *The function* rank *satisfies*
  *(R1) If $X \subseteq E$, then $0 \leq \mathrm{rank}(X) \leq |X|$.*
  *(R2) If $X \subseteq Y \subseteq E$, then $\mathrm{rank}(X) \leq \mathrm{rank}(Y)$.*
  *(R3) If $X$ and $Y$ are subsets of $E$ then*

$$\mathrm{rank}(X \cup Y) + \mathrm{rank}(X \cap Y) \leq \mathrm{rank}(X) + \mathrm{rank}(Y).$$

One can easily get the other alternative characterisations of a matroid via the rank function as follows:

6.1.7. LEMMA. *Let $\mathsf{M}$ be a matroid with rank function* rank *and let $X \subseteq E(M)$. Then*

- *$X$ is independent if and only if $|X| = \mathrm{rank}(X)$.*
- *$X$ is a basis if and only if $|X| = \mathrm{rank}(X) = \mathrm{rank}(\mathsf{M})$.*
- *$X$ is a circuit if and only if $X$ is non-empty and for all $x \in X$, we have $\mathrm{rank}(X \setminus \{x\}) = |X| - 1 = \mathrm{rank}(X)$.*

The last way to characterise a matroid we introduce is via closed sets, also known as flats.

6.1.8. DEFINITION. Let $\mathsf{M}$ be a matroid with rank funktion rank and ground set $E$. We say a subset $F \subseteq E$ is *closed*, or a *flat*, if for all $x \in E \setminus F$ we have

$$\mathrm{rank}(F \cup \{x\}) = \mathrm{rank}(F) + 1.$$

6.1.9. THEOREM. *Let $E$ be a finite ground set and let $\mathcal{F}$ be a collection of subsets of $E$. The following are equivalent:*

- *$(E, \mathcal{I})$ is a matroid that has $\mathcal{F}$ as set of flats.*
- *$\mathcal{F}$ satisfies:*
  *(F1) $E \in \mathcal{F}$.*
  *(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.*
  *(F3) If $F \in \mathcal{F}$ and $\{F_1, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ that properly contain $F$, then the sets $F_1 \setminus F$, $F_2 \setminus F$, $\ldots$, $F_k \setminus F$ partition $E \setminus F$.*

6.1.10. Definition. Let $\mathsf{M}$ be a matroid. A *loop* of $\mathsf{M}$ is an element $x \in E(\mathsf{M})$ such that $\{x\}$ is dependent. Two non-loop elements $x, y \in E(\mathsf{M})$ are *parallel* if $\{x, y\}$ is dependent. The matroid $\mathsf{M}$ is *simple* if it does not have loops or parallel elements.

Since matroids are a combinatorial abstraction of dependence coming from linear algebra and graph theory, it is useful to understand how the above terms are interpreted in these contexts.

6.1.11. Example.

- The linear-algebraic model for a matroid is derived from a list of vectors $X = (x_i : i \in E)$, indexed by a finite set $E$, that span a $r$-dimensional vector space $V$. The matroid of $X$, denoted $\mathsf{M}[X]$, is the matroid whose ground set is $E$ and

  $\mathcal{B}(\mathsf{M}[X]) = \{B \subseteq E : (x_i : i \in B) \text{ is a vector space basis of } V\}$.

  The independent sets of $\mathsf{M}[X]$ correspond to collections of linearly independent vectors in $X$, the rank function of $\mathsf{M}[X]$ records the dimension of the linear span of the input vectors, and the flats of $\mathsf{M}[X]$ are the subsets of $E$ such that the linear span of the corresponding vectors contains no other elements of $X$. Given a field $\mathcal{F}$, a matroid $\mathsf{M}$ is $\mathcal{F}$-realisable if $\mathsf{M} = \mathsf{M}[X]$ for a $E(\mathsf{M})$-indexed list $X$ of vectors in a $\mathcal{F}$-vector space. When $x_i \neq 0$ for each $i \in E(\mathsf{M})$, the vectors $x_i$ define points in the projective space $\mathbb{P}(V)$; this is called a *projective realization* of $\mathsf{M}[X]$.
- Circuits are best understood from the graph-theoretic viewpoint. Let $\Gamma$ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The matroid of $\Gamma$, denoted $\mathsf{M}[\Gamma]$, is the matroid with ground set $E = E(\Gamma)$ and the bases $\mathcal{B}(\mathsf{M}[\Gamma])$ consist of the edge sets of the maximal forests of $\Gamma$. A circuit of $\mathsf{M}[\Gamma]$ is a subset of edges that form a cycle of $\Gamma$.

6.1.12. Definition. Given two matroids $\mathsf{M}_1$ and $\mathsf{M}_2$, an *isomorphism* of $\mathsf{M}_1$ and $\mathsf{M}_2$ is a bijection $\varphi : E(\mathsf{M}_1) \to E(\mathsf{M}_2)$ such that $A \in \mathcal{B}(\mathsf{M}_1)$ if and only if $\varphi(A) \in \mathcal{B}(\mathsf{M}_2)$. An *automorphism* of a matroid $\mathsf{M}$ is an isomorphism from $\mathsf{M}$ to itself. The group (under function composition) of all automorphisms of $\mathsf{M}$ is called the *automorphism group* of $\mathsf{M}$ and is denoted by $\mathrm{Aut}(\mathsf{M})$.

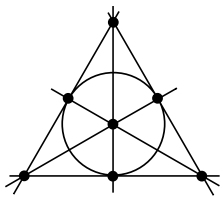


FIGURE 1. A projective realization over $\mathcal{F}_2$ of the Fano matroid $\mathsf{F}$

6.1.13. EXAMPLE (Uniform matroid). The rank-$r$ *uniform matroid* on the finite set $E$, denoted $\mathsf{U}_{r,E}$ is the matroid with ground set $E$ and whose bases are the $r$-element subsets of $E$. When $E = \{1, \dots, n\}$, we simply write $\mathsf{U}_{r,n}$. This is the matroid of a collection of $n$ vectors in $V$ in linear general position, where $V$ is a $r$-dimensional vector space over a field of infinite order. The automorphism group of $\mathsf{U}_{r,E}$ is the symmetric group $S_E$.

6.1.14. EXAMPLE (Fano matroid). A fundamental example is the Fano matroid defined by $\mathsf{F} = \mathsf{M}[X]$ where $X$ is the sequence of vectors in $(\mathcal{F}_2)^3$ given by the columns of the matrix (which we also denote by $X$)

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

These are exactly the seven nonzero vectors in $(\mathcal{F}_2)^3$. The automorphism group of $\mathsf{F}$ is isomorphic to $\mathrm{PSL}_3(\mathcal{F}_2)$. This matroid is significant because it is the smallest example which is not $\mathbb{C}$-realisable.

The automorphism group of a graph $\Gamma$, denoted by $\mathrm{Aut}(\Gamma)$, differs from $\mathrm{Aut}(\mathsf{M}[\Gamma])$ in significant ways. The former is a subgroup of $S_{V(\Gamma)}$, whereas the latter is a subgroup of $S_{E(\Gamma)}$. To provide a direct comparison, we may consider the action of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$, which defines a group homomorphism $\mathrm{Aut}(\Gamma) \to S_{E(\Gamma)}$. The image, denoted by $\mathrm{Aut}_E(\Gamma)$, is a subgroup of $S_{E(\Gamma)}$. Nevertheless, $\mathrm{Aut}_E(\Gamma)$ and $\mathrm{Aut}(\mathsf{M}[\Gamma])$ need not be the same subgroup of $S_{E(\Gamma)}$. The difference between these two groups is the content of Whitney's 2-isomorphism theorem [87]; see also [67, Theorem 5.3.1].

## 6.2. Quantum Automorphisms Associated to Different Axiom Systems

We begin with a general procedure for producing quantum subgroups of $S_E^+$, where $E$ is a finite set. Denote by $\mathrm{Tup}(E)$ the set of tuples of elements of $E$ that have length at most $|E|$. Given tuples $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ of the same length, define

$$u_{AB} = u_{a_1,b_1} \cdots u_{a_k,b_k}.$$

We gather some useful properties of the terms $u_{AB}$ in $S_E^+$.

6.2.1. Proposition. *Suppose $A, B \in E^k$. The following equalities hold in $S_E^+$.*

*(i) We have*

$$\sum_{C \in E^k} u_{AC} = 1 \quad \textit{and} \quad \sum_{C \in E^k} u_{CB} = 1.$$

*(ii) The coproduct of $u_{AB}$ is*

$$\Delta(u_{AB}) = \sum_{C \in E^k} u_{AC} \otimes u_{CB}$$

Proof. To prove (i), by symmetry, it suffices to prove just the left equality. We proceed by induction on $k$. The base case $k = 1$ follows from the definition of $S_E^+$. Let $A = (a_1, \ldots, a_{k-1}, a)$ and $A' = (a_1, \ldots, a_{k-1})$. Then

$$\sum_{C \in E^k} u_{AC} = \sum_{C' \in E^{k-1}} u_{A'C'} \sum_{c \in E} u_{ac}$$

Both sums on the right equal 1, the former by the inductive hypothesis and the latter by the base case.

For (ii), we again proceed by induction on $k$. The base case $k = 1$ is just the coproduct formula. Let $A, A'$ be as in the proof of (i) and let $B = (b_1, \ldots, b_{k-1}, b)$, $B' = (b_1, \ldots, b_{k-1})$. Then

$$\Delta(u_{AB}) = \Delta(u_{A'B'})\Delta(u_{ab}) = \left(\sum_{C' \in E^{k-1}} u_{A'C'} \otimes u_{C'B'}\right)\left(\sum_{c \in E} u_{ac} \otimes u_{cb}\right) = \sum_{C \in E^k} u_{AC} \otimes u_{CB}.$$

The second equality follows from the inductive hypothesis. □

Fix a nonempty subset $\mathcal{A} \subseteq \mathrm{Tup}(E)$. Define the ideal $I_{\mathcal{A}}$ of $S_E^+$ by

$$I_{\mathcal{A}} = \langle u_{AB} \,:\, (A \in \mathcal{A} \text{ and } B \notin \mathcal{A}) \text{ or } (A \notin \mathcal{A} \text{ and } B \in \mathcal{A}) \rangle. \tag{6.2.1}$$

Let $G_{\mathcal{A}} = S_E^+ / I_{\mathcal{A}}$.

6.2.2. Proposition. *If $I_{\mathcal{A}}$ is self-adjoint, then the quotient $G_{\mathcal{A}}$ is a subgroup of the quantum symmetric group $S_E^+$.*

Proof. It suffices to show that the coproduct $\Delta$ on $S_E^+$ restricts to a coproduct on $G_{\mathcal{A}}$. By Proposition 6.2.1 we have

$$\Delta(u_{AB}) = \sum_{C \in E^k} u_{AC} \otimes u_{CB}.$$

Suppose $u_{AB} \in I_{\mathcal{A}}$. Without loss of generality, we may assume that $A \in \mathcal{A}$ and $B \notin \mathcal{A}$. If $C \in \mathcal{A}$, then $u_{CB} \in I_{\mathcal{A}}$. Otherwise, $C \notin \mathcal{A}$, and so $u_{AC} \in I_{\mathcal{A}}$. Therefore, each summand of $\Delta(u_{AB})$ lies in $(I_{\mathcal{A}} \otimes S_E^+) + (S_E^+ \otimes I_{\mathcal{A}})$, as required. □

Let $\mathsf{M}$ be a rank-$r$ matroid, and let $A$ be a tuple. Then $A$ is:

- an *independent tuple* if it has no repeated elements and its underlying set is independent;
- a *dependent tuple* if it is not an independent tuple;

- a *basis tuple* if it is independent and has length $r$;
- a *flat tuple* if it has no repeated elements and its underlying set is a flat of $\mathsf{M}$;
- a *circuit tuple* if $A = (a, a)$ for a nonloop $a$ or if $A$ has no repeating elements and its underlying set is a circuit of $\mathsf{M}$.

Denote by $\overline{\mathcal{I}}(\mathsf{M})$, $\overline{\mathcal{B}}(\mathsf{M})$, $\overline{\mathcal{F}}(\mathsf{M})$, and $\overline{\mathcal{C}}(\mathsf{M})$ the sets of independent, basis, flat, and circuit tuples of $\mathsf{M}$, respectively. Since the corresponding ideals $I_{\overline{\mathcal{I}}(\mathsf{M})}$, $I_{\overline{\mathcal{B}}(\mathsf{M})}$, $I_{\overline{\mathcal{F}}(\mathsf{M})}$,$I_{\overline{\mathcal{C}}(\mathsf{M})}$ are self-adjoint, we may apply Proposition 6.2.2 to these sets to define quantum automorphism groups of matroids.

6.2.3. Definition. Let $\mathsf{M}$ be a matroid.

- The *independent sets* quantum automorphism group is $G^{\mathcal{I}}_{aut}(\mathsf{M}) = G_{\overline{\mathcal{I}}(\mathsf{M})}$.
- The *bases* quantum automorphism group is $G^{\mathcal{B}}_{aut}(\mathsf{M}) = G_{\overline{\mathcal{B}}(\mathsf{M})}$.
- The *flats* quantum automorphism group is $G^{\mathcal{F}}_{aut}(\mathsf{M}) = G_{\overline{\mathcal{F}}(\mathsf{M})}$.
- The *circuits* quantum automorphism group is $G^{\mathcal{C}}_{aut}(\mathsf{M}) = G_{\overline{\mathcal{C}}(\mathsf{M})}$.

Throughout this chapter, we will write $I(G)$ to denote the ideal defining the quantum permutation group $G$, so for example we will write $I(G^{\mathcal{I}}_{aut}(\mathsf{M}))$ instead of $I_{\overline{\mathcal{I}}(\mathsf{M})}$.

Given a quantum permutation group $G \subseteq S_E^+$, denote by $G^{\mathrm{com}}$ the commutative quantum permutation group

$$G^{\mathrm{com}} = G/\langle u_{ab}u_{cd} - u_{cd}u_{ab} \,:\, a, b, c, d \in E\rangle.$$

The following proposition verifies that the quantum automorphism groups in Definition 6.2.3 are, a priori, quantisations of $\mathrm{Aut}(\mathsf{M})$.

6.2.4. Proposition. *The commutative quantum groups*

$$G^{\mathcal{I}}_{aut}(\mathsf{M})^{\mathrm{com}},\ G^{\mathcal{B}}_{aut}(\mathsf{M})^{\mathrm{com}},\ G^{\mathcal{F}}_{aut}(\mathsf{M})^{\mathrm{com}},\ G^{\mathcal{C}}_{aut}(\mathsf{M})^{\mathrm{com}}$$

*are all isomorphic to* $C(\mathrm{Aut}(\mathsf{M}))$.

Before proving this proposition, we establish some notation. Given a finite set $E$, denote by $\mathrm{End}(\mathbb{C}^E)$ the ring of linear maps $\mathbb{C}^E \to \mathbb{C}^E$. We view the symmetric group $S_E$ as the multiplicative subgroup of permutation matrices in $\mathrm{End}(\mathbb{C}^E)$. Given $\sigma \in S_E$ and $i, j \in E$, denote by $\sigma_{ij}$ the $(i, j)$–entry of the permutation matrix associated to $\sigma$, and define the homomorphism $\varphi_\sigma : S_E^+ \to \mathbb{C}$ by $\varphi_\sigma(u_{ij}) = \sigma_{ij}$.

Proof of Proposition 6.2.4. We focus on the case $G^{\mathcal{I}}_{aut}(\mathsf{M})$, as the others are similar. Let $G = G^{\mathcal{I}}_{aut}(\mathsf{M})^{\mathrm{com}}$. By [37, Exercise 1.10], $G \cong C(\mathcal{G})$ where $\mathcal{G} \subseteq S_E$ is the permutation group

$$\mathcal{G} = \{\sigma \in S_E \,:\, \varphi_\sigma(f) = 0 \text{ for all } f \in I(G)\}.$$

We claim that $\mathcal{G} = \mathrm{Aut}(\mathsf{M})$. Given tuples $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$, we have

$$\varphi_\sigma(u_{AB}) = \sigma_{a_1 b_1} \cdots \sigma_{a_k b_k} = \begin{cases} 1 & \text{if } \sigma(A) = B, \\ 0 & \text{if } \sigma(A) \neq B. \end{cases} \tag{6.2.2}$$

Suppose $\sigma \in \mathrm{Aut}(\mathsf{M})$ and $f \in I(G)$ – without loss of generality we may assume that $f = u_{AB}$ where $A$ is independent and $B$ is dependent. Since $\sigma$ is an automorphism of $\mathsf{M}$, $\sigma(A)$ cannot equal $B$, and so $\varphi_\sigma(u_{AB}) = 0$ by Equation (6.2.2). Conversely, suppose $\sigma \in \mathcal{G}$. If $A$ is independent, then $\sigma(A) \neq B$ for all dependent tuples $B$ by Equation (6.2.2) and the definition of $I(G)$, and so $\sigma(A)$ is independent. Similarly, if $D = \sigma(A)$ is independent, then $\sigma^{-1}(D) \neq C$ for all dependent tuples $C$ (since $\sigma^{-1} \in \mathcal{G}$), and so $A$ is independent. Therefore $\sigma \in \mathrm{Aut}(\mathsf{M})$, as required. □

For the remainder of this section, we describe the relationship between the various quantum automorphism groups of matroids.

6.2.5. THEOREM. *For any matroid* $\mathsf{M}$,

$$G^{\mathcal{I}}_{aut}(\mathsf{M}) = G^{\mathcal{B}}_{aut}(\mathsf{M}).$$

We begin with the following technical lemma. Given a tuple $A = (a_1, \ldots, a_\ell)$ of length $\ell$ an element $x \in E$, and $1 \leq i \leq \ell + 1$, define $A \cup_i x$ to be the tuple of length $\ell + 1$ obtained by inserting $x$ at position $i$, i.e.,

$$A \cup_i x = (a_1, \ldots, a_{i-1}, x, a_i, \ldots, a_\ell).$$

If $i = 1$ then $A \cup_i x$ is obtained by prepending $x$ to $A$, and if $i = \ell + 1$ then $A \cup_i x$ is obtained by appending $x$ to the end of $A$.

6.2.6. LEMMA. *Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathrm{Tup}(E)$. Suppose that, for each pair of tuples $A = (a_1, \ldots, a_\ell)$, $B = (b_1, \ldots, b_\ell)$ of the same length with $A \in \mathcal{A}_2$ and $B \notin \mathcal{A}_2$, there is a $x \in E$ and $i \in \{1, \ldots, \ell\}$ such that $A \cup_i x \in \mathcal{A}_1$, but $B \cup_i y \notin \mathcal{A}_1$ for all $y \in E$. Then $G_{\mathcal{A}_1} \subseteq G_{\mathcal{A}_2}$.*

PROOF. We must show that $I_{\mathcal{A}_2} \subseteq I_{\mathcal{A}_1}$. Suppose $u_{AB} \in I_{\mathcal{A}_2}$, without loss of generality, we may assume that

$$A = (a_1, \ldots, a_\ell) \in \mathcal{A}_2 \quad \text{and} \quad B = (b_1, \ldots, b_\ell) \notin \mathcal{A}_2.$$

Let $x \in E$ and $1 \leq i \leq \ell$ be as in the lemma, and set $A' = (a_1, \ldots, a_{i-1})$, $A'' = (a_i, \ldots, a_\ell)$, $B' = (b_1, \ldots, b_{i-1})$, and $B'' = (b_i, \ldots, b_\ell)$. By the relation $\sum_{y \in E} u_{xy} = 1$, we have

$$u_{AB} = \sum_{y \in E} u_{A'B'} u_{xy} u_{A''B''}$$

By hypothesis, each term in this sum lies in $I_{\mathcal{A}_1}$, and therefore so does $u_{AB}$, as required. □

Proof of Theorem 6.2.5. For $1 \leq k \leq \operatorname{rank}(\mathsf{M})$, denote by $\overline{\mathcal{I}}_k$ the set of independent tuples of length $k$. Let now $k < \operatorname{rank}(\mathsf{M})$ and suppose $A \in \overline{\mathcal{I}}_k(\mathsf{M})$ and $B \in E(\mathsf{M})^k \setminus \overline{\mathcal{I}}_k$. Because $A$ is independent and not a basis, there is a $x \in E(\mathsf{M}) \setminus A$ such that the length–$(k+1)$ tuple $A \cup_{k+1} x$ is independent. Furthermore, because $B$ is dependent, the length–$(k+1)$ tuple $B \cup_{k+1} y$ is also dependent. By Lemma 6.2.6, we have that $G_{\overline{\mathcal{I}}_{k+1}} \subseteq G_{\overline{\mathcal{I}}_k}$. Thus, we get the chain

$$G^{\mathcal{B}}_{aut}(\mathsf{M}) = G_{\overline{\mathcal{I}}_r} \subseteq G_{\overline{\mathcal{I}}_{r-1}} \subseteq \cdots \subseteq G_{\overline{\mathcal{I}}_1}.$$

where $r = \operatorname{rank}(\mathsf{M})$, and hence $G^{\mathcal{B}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{I}}_{aut}(\mathsf{M})$. The inclusion $G^{\mathcal{I}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{B}}_{aut}(\mathsf{M})$ follows from $\overline{\mathcal{B}}(\mathsf{M}) \subseteq \overline{\mathcal{I}}(\mathsf{M})$. $\square$

Next, we compare $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ for simple matroids $\mathsf{M}$ (i.e., without loops or parallel elements) with rank 3. We define a *triangle* to be a triple of three rank-2 flats $\{F_1, F_2, F_3\}$ that pairwise intersect but $F_1 \cap F_2 \cap F_3 = \emptyset$.

6.2.7. Theorem. *Suppose* $\mathsf{M}$ *is a simple, rank* 3 *matroid. If* $E(\mathsf{M}) \neq F_1 \cup F_2 \cup F_3$ *for all triangles* $\{F_1, F_2, F_3\}$*, then*

$$G^{\mathcal{C}}_{aut}(\mathsf{M}) \subseteq G^{\mathcal{B}}_{aut}(\mathsf{M}).$$

In the proof of this theorem, we make use of the *closure* operator of a matroid $\mathsf{M}$. This is the function $\operatorname{cl}_{\mathsf{M}} : 2^{E(\mathsf{M})} \to \mathcal{F}(\mathsf{M})$ defined by

$$\operatorname{cl}_{\mathsf{M}}(A) = \{x \in E(\mathsf{M}) \,:\, \operatorname{rank}_{\mathsf{M}}(A \cup \{x\}) = \operatorname{rank}_{\mathsf{M}}(A)\}.$$

Proof of Theorem 6.2.7. We must show that $I(G^{\mathcal{B}}_{aut}(\mathsf{M})) \subseteq I(G^{\mathcal{C}}_{aut}(\mathsf{M}))$. Suppose $A = (a_1, a_2, a_3)$ is a basis tuple and $B = (b_1, b_2, b_3)$ is dependent. If $B$ is a circuit tuple, then $u_{AB} \in I(G^{\mathcal{C}}_{aut}(\mathsf{M}))$. Suppose that $B$ is not a circuit. Because $\mathsf{M}$ is simple, the tuple $B$ must have repeated elements. If $b_1 = b_2$ or $b_2 = b_3$, then $u_{AB} \in I(S^+_E) \subseteq I(G^{\mathcal{C}}_{aut}(\mathsf{M}))$.

Finally, consider the case $b_1 = b_3$. Let $F_1 = \operatorname{cl}(\{a_2, a_3\})$, $F_2 = \operatorname{cl}(\{a_1, a_3\})$, and $F_3 = \operatorname{cl}(\{a_1, a_2\})$. If there was an element $f \in F_1 \cap F_2 \cap F_3$, then we would have $\operatorname{rank}(\{a_1, f\}) = 2$, since there are no parallel elements. However, since $f \in \operatorname{cl}(\{a_1, a_2\})$, we get that $\operatorname{rank}(\{a_1, a_2, f\}) = 2$ and thus $a_2 \in \operatorname{cl}(\{a_1, f\})$. Similarly, we get $a_3 \in \operatorname{cl}(\{a_1, f\})$. But then we would have $\{a_1, a_2, a_3\} \subseteq \operatorname{cl}(\{a_1, f\})$, which is a contradiction to $\operatorname{rank}(\{a_1, f\}) = 2$ since $\operatorname{rank}(\{a_1, a_2, a_3\}) = 3$, and therefore $F_1 \cap F_2 \cap F_3$ must be empty.

We thus see that $\{F_1, F_2, F_3\}$ is a triangle and by hypothesis, we get that there is a $a_4 \in E(\mathsf{M}) \setminus (F_1 \cup F_2 \cup F_3)$, and therefore $(a_1, a_2, a_3, a_4)$ is a circuit tuple. So

$$u_{AB} = u_{a_1 b_1} u_{a_2 b_2} u_{a_3 b_1} = \sum_{x \in E} u_{a_1 b_1} u_{a_2 b_2} u_{a_3 b_1} u_{a_4 x}.$$

Because $(b_1, b_2, b_1, x)$ is not a circuit tuple for each $x \in E$, each summand on the right is in $I(G^{\mathcal{C}}_{aut}(\mathsf{M}))$, as required. $\square$

In Section 6.5, we list several examples where $G^{\mathcal{C}}_{aut}(\mathsf{M})$ is a proper subgroup of $G^{\mathcal{B}}_{aut}(\mathsf{M})$. Nevertheless, we have not found any examples where, conversely, $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is a proper subgroup of $G^{\mathcal{C}}_{aut}(\mathsf{M})$. This raises the following question.

6.2.8. QUESTION. Does the conclusion to Theorem 6.2.7 hold for every matroid?

At any rate, the lattice of flat axioms never generate proper quantum automorphisms:

6.2.9. THEOREM. *For any matroid* $\mathsf{M}$*, we have*

$$G^{\mathcal{F}}_{aut}(\mathsf{M}) = C(\mathrm{Aut}(\mathsf{M})).$$

Suppose $N = |E(\mathsf{M})|$. Because $u_{AB} \in I(G^{\mathcal{F}}_{aut}(\mathsf{M}))$ for $A, B \in E(\mathsf{M})^N$ such that $A$ does not have repeated elements and $B$ has repeated elements, this theorem is a direct consequence of Proposition 6.2.4 and the following general lemma.

6.2.10. LEMMA. *Suppose $E$ is a set with $N$ elements and $\mathcal{A} \subseteq \mathrm{Tup}(E)$. If $u_{AB} = 0$ in $G_{\mathcal{A}}$ for all $A, B \in E^N$ such that $A$ has no repeated elements and $B$ does have repeated elements, then $G_{\mathcal{A}}$ is commutative.*

PROOF. Let $0 \leq k \leq N-1$. First, we claim that $u_{AB} = 0$ in $G_{\mathcal{A}}$ for all $A, B \in E^{N-k}$ such that $A$ has no repeated elements and $B$ does have repeated elements. We proceed by induction on $k$, and the base case $k=0$ is a hypothesis of the lemma. For $k \geq 1$, suppose $A$ and $B$ are as above and have $N-k$ elements. If $a \notin A$ then

$$u_{AB} = \sum_{b \in E} u_{AB} u_{ab}.$$

As each summand on the right lies in $I_{\mathcal{A}}$ by the inductive hypothesis, we also have $u_{AB} = 0$ in $G_{\mathcal{A}}$, as required.

Now, suppose $a, b, c, d \in E$ and consider $u_{ab}u_{cd}$. If $a = c$ or $b = d$, then $u_{ab}u_{cd}$ and $u_{cd}u_{ab}$ both equal 0 or both equal $u_{ab}$, and so $u_{ab}u_{cd} = u_{cd}u_{ab}$. Assume $a \neq c$ and $b \neq d$. Then

$$u_{ab}u_{cd} = \sum_{x \in E} u_{ab}u_{cd}u_{xb}.$$

For $x \neq a, c$, the summand $u_{ab}u_{cd}u_{xb}$ lies in $I_{\mathcal{A}}$ by the above claim, and summand $u_{ab}u_{cd}u_{cb}$ equals 0 by the quantum symmetric group relations. Therefore

$$u_{ab}u_{cd} = u_{ab}u_{cd}u_{ab}.$$

As the term on the right is self-adjoint, we have $u_{ab}u_{cd} = u_{cd}u_{ab}$, and so $G_{\mathcal{A}}$ is commutative, as required. □

## 6.3. Quantum Symmetries for Matroids by Rank and Girth

We begin this section by developing general descriptions of the quantum automorphism group $G^{\mathcal{B}}_{aut}(\mathsf{M})$ for matroids of rank 1 and 2. Given a matroid $\mathsf{M}$ and

$L \subseteq E(\mathsf{M})$, the *deletion* of $L$ from $\mathsf{M}$ is the matroid $\mathsf{M} \setminus L$ with ground set $E(\mathsf{M}) \setminus L$ and whose independent sets are $\mathcal{I}(\mathsf{M} \setminus L) = \{A \in \mathcal{I}(\mathsf{M}) : A \subseteq E(\mathsf{M}) \setminus L\}$.

6.3.1. PROPOSITION. *Let* $\mathsf{M}$ *be a matroid and denote by* $L \subseteq E(\mathsf{M})$ *the set of loops of* $\mathsf{M}$. *Then*

$$G_{aut}^{\mathcal{I}}(\mathsf{M}) \cong S_L^+ * G_{aut}^{\mathcal{I}}(\mathsf{M} \setminus L)$$

*If* $\mathsf{M}$ *has rank 1, then*

$$G_{aut}^{\mathcal{I}}(\mathsf{M}) \cong S_L^+ * S_{E(\mathsf{M})\setminus L}^+.$$

PROOF. Recall that $i \in E(\mathsf{M})$ is a loop of $\mathsf{M}$ if $\{i\}$ is a dependent set. So this proposition follows from the fact that $u_{ij} \in I(G_{aut}^{\mathcal{I}}(\mathsf{M}))$ if $i \in L$ and $j \in E(\mathsf{M}) \setminus L$, or $i \in E(\mathsf{M}) \setminus L$ and $j \in L$. ☐

Now suppose that $\mathsf{M}$ is a loopless rank 2 matroid. The rank 1 flats form a partition of $E(\mathsf{M})$. Define a graph $\Gamma_2[\mathsf{M}]$ in the following way. Let $V(\Gamma_2[\mathsf{M}]) = E(\mathsf{M})$, and two vertices are connected by an edge if they lie in different rank-1 flats.

6.3.2. PROPOSITION. *Let* $\mathsf{M}$ *be a rank 2 matroid and let* $L \subseteq E(\mathsf{M})$ *be its set of loops. Then*

$$G_{aut}^{\mathcal{B}}(\mathsf{M}) \cong S_L^+ * \ G_{aut}(\Gamma_2[\mathsf{M} \setminus L]).$$

PROOF. Given a graph $\Gamma$, by [77, Lemma 5.7] its quantum automorphism group is the subgroup of $S_{V(\Gamma)}^+$ defined by the ideal

$$\langle u_{ab}u_{cd} : (ac \in E(\Gamma) \text{ and } bd \notin E(\Gamma)) \text{ or } (ac \notin E(\Gamma) \text{ and } bd \in E(\Gamma))\rangle$$

If $\mathsf{M}$ is loopless, then the proposition follows from this description of $G_{aut}(\Gamma_2[\mathsf{M}])$ and the definition of $G_{aut}^{\mathcal{B}}(\mathsf{M})$. For the general case, apply Proposition 6.3.1. ☐

Therefore, $G_{aut}^{\mathcal{B}}(\mathsf{M})$ are familiar quantum automorphism groups for matroids of rank 1 and 2. At the other end of the rank spectrum, we develop a criterion which guarantees that $G_{aut}^{\mathcal{B}}(\mathsf{M})$ is commutative for large classes of matroids of rank $\geq 4$. The *girth* of the matroid $\mathsf{M}$, denoted girth($\mathsf{M}$), is the size of the smallest circuit of $\mathsf{M}$.

6.3.3. THEOREM. *If* $\mathsf{M}$ *is a matroid with* girth($\mathsf{M}$) $\geq 4$, *then*

$$G_{aut}^{\mathcal{B}}(\mathsf{M}) = G_{aut}^{\mathcal{I}}(\mathsf{M}) = C(\mathrm{Aut}(\mathsf{M})).$$

A rank $r$ matroid is *paving* if its girth is $\geq r$. Conjecturally, almost all matroids are paving [27], [62], and paving matroids of rank $\geq 4$ have commutative bases quantum automorphism group by Theorem 6.3.3. This leaves rank $r = 3$, and there are several examples of matroids $\mathsf{M}$ such that $G_{aut}^{\mathcal{B}}(\mathsf{M})$ is noncommutative, see §6.4.

PROOF OF THEOREM 6.3.3. In view of Theorem 6.2.5, we may focus on proving the second equality in the theorem. By Proposition 6.2.4, it suffices to show that, for a matroid $\mathsf{M}$ with girth($\mathsf{M}$) $\geq 4$, we have

$$u_{ab}u_{cd}u_{ab} = u_{ab}u_{cd} \text{ in } G^{\mathcal{I}}_{aut}(\mathsf{M})$$

for all $a, b, c, d \in E(\mathsf{M})$. Indeed, this formula implies that $u_{ab}$ and $u_{cd}$ commute since $u_{ab}u_{cd}u_{ab}$ is self-adjoint.

**Case 1**. If $a = c$ and $b = d$, then

$$u_{ab}u_{cd}u_{ab} = u_{ab}^2 u_{ab} = u_{ab}u_{cd}.$$

**Case 2**. If $a \neq c$ and $b = d$ then

$$u_{ab}u_{cd}u_{ab} = u_{ab}u_{cb}u_{ab} = 0 = u_{ab}u_{cb}.$$

**Case 3**. If $a = c$ and $b \neq d$, argue as in Case 2.

**Case 4**. Suppose $a \neq c$ and $b \neq d$. Then

$$u_{ab}u_{cd} = u_{ab}u_{cd}\sum_{x\in E} u_{ax}.$$

We must show that $u_{ab}u_{cd}u_{ax} = 0$ in $G^{\mathcal{I}}_{aut}(\mathsf{M})$ for every $x \in E(\mathsf{M})\setminus\{b\}$. Since $(a, c, a)$ has repeated elements, it is not an independent tuple. On the other hand, $(b, d, x)$ is an independent tuple for $x \neq b, d$ by the hypothesis girth($\mathsf{M}$) $\geq 4$. If $x = d$, then

$$u_{ab}u_{cd}u_{ax} = u_{ab}u_{cd}u_{ad} = 0$$

since $a \neq c$. In either case, we have

$$u_{ab}u_{cd}u_{ax} = 0 \text{ in } G^{\mathcal{I}}_{aut}(\mathsf{M}) \text{ for } x \in E(\mathsf{M}) \setminus \{b\}$$

as required. □

## 6.4. Computational Results

The study of the quantum automorphism groups of a given matroid, beyond the simplest cases, is impractical to do by hand. Therefore, we rely on computer algebra to study specific examples. Our primary goal is to determine, for as many matroids $\mathsf{M}$ as possible, whether $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are commutative, which in view of Proposition 6.2.4 is equivalent to being isomorphic to $C(\mathrm{Aut}(\mathsf{M}))$. In view of Remark 1.2.7, and since the coproduct formula remains valid over $\mathbb{Z}$, it suffices to compute with quantum permutation groups with rational coefficients.

We employ the free open-source software `OSCAR` [28], [66] which features two different methods for computing noncommutative Gröbner bases. The results are given as tables in Section 6.5.

**6.4.1. Algorithms and their Implementations.** Hilbert's basis theorem says that every ideal $I$ in a commutative polynomial ring is finitely generated, i.e., such a ring is *Noetherian*. Furthermore, the ideal $I$ has particularly useful generating systems, known as *Gröbner bases*, which, for example, allow to decide whether a given polynomial is contained in $I$ or not. Buchberger's algorithm is a classical method to convert any finite generating system of $I$ into a Gröbner basis. For details, we recommend the textbook by von zur Gathen and Gerhard [40, §21].

Our computations employ a noncommutative generalisation of Buchberger's algorithm, which has already been introduced in Section 3.1. In our computations we always used the degree reverse lexicographic ordering. We consider the polynomial ring $R = \mathbb{Q}\langle X \rangle$ with a finite set $X$ of noncommuting variables and rational coefficients. The biggest obstacle to overcome for *any* algorithm dealing with ideals of $R$ is the fact that $R$ is not Noetherian.

Since $R$ is not Noetherian, the Buchberger algorithm may not terminate, but if it does, then the output is a *Gröbner basis* of $I$, with respect to the degree reverse lexicographic ordering. Term orderings are subtle in the noncommutative case, whence we stick to this particular term ordering throughout. In addition to the basic algorithm, we use some simple optimizations to make the computation run faster. The first step for this is to make the set of obstructions $B$ into a priority queue that is sorted in the degree reverse lexicographic order, which is an improvement discussed in the PhD thesis of Keller [45]. Moreover, for computing the obstructions we use the Aho–Corasick algorithm [1] for efficiently finding substrings in a given text to improve the function that checks for divisibility by the partial Gröbner basis. This works by maintaining an Aho–Corasick automaton during the computation that contains all elements of the Gröbner basis computed so far. When checking for a given monomial whether it is divisible by an element of the Gröbner basis, one looks for the first element in the automaton that matches a substring of the monomial. If such an element exists, that monomial is divisible by the partial Gröbner basis. The noncommutative Buchberger algorithm 3.1.19 we used was implemented in `OSCAR` directly.

La Scala and Levandovskyy proposed a different way of dealing with not finitely generated ideals [47]. To the generating system $\mathcal{G}$ of an ideal $I$ in the noncommutative polynomial ring $R$ they associate an ideal, called the *letterplace ideal* of $I$, which lives in a commutative polynomial ring, but with infinitely many variables. By restricting to subrings with finitely many variables, say $d$, this method allows to employ standard commutative Gröbner bases to obtain *truncated Gröbner bases* of $I$. The number $d$ is called the *degree bound* of the truncation, and it is part of the input.

In contrast with the Buchberger algorithm sketched in Section 3.1, the advantage of the Letterplace algorithm is its termination for every input. The drawback is that it does not directly yield a Gröbner basis even if it exists. On the bright side, we have the following key result.

6.4.1. Theorem (La Scala and Levandovskyy [47, Cor. 3.19]). *Let $I \subseteq R$ be a graded two-sided ideal with a finite homogeneous basis whose polynomials are all* *of degree d. Denote by $\mathcal{G}_{d-1}$ the truncated Gröbner basis of $I$ up to degree d. If $\mathcal{G}_{d-1} = \mathcal{G}_{2d-2}$, then $\mathcal{G}_{d-1}$ is a Gröbner basis of $I$.*

That is to say, if the sequence of truncated Gröbner bases stabilizes enough, then the final truncated Gröbner basis is actually a proper Gröbner basis. This method is implemented in `Letterplace` [49], which is also available in `OSCAR`.

Either method for computing noncommutative Gröbner bases leads to a procedure for semi-deciding the commutativity of the quantum automorphism group $G^{\mathcal{B}}_{aut}(\mathsf{M})$. Given a Gröbner basis of the ideal $I(G^{\mathcal{B}}_{aut}(\mathsf{M}))$ we can check if all commutators $u_{ij}u_{k\ell} - u_{k\ell}u_{ij}$ lie in $I(G^{\mathcal{B}}_{aut}(\mathsf{M}))$. Similarly for $G^{\mathcal{C}}_{aut}(\mathsf{M})$. If the Gröbner basis computation does not terminate, we cannot say anything. Nevertheless, using these techniques we are able to determine if $G^{\mathcal{B}}_{aut}(\mathsf{M})$ or $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are commutative for numerous matroids of small rank and ground set, and so we pose the following question.

6.4.2. Question. Is commutativity of $G^{\mathcal{B}}_{aut}(\mathsf{M})$ or $G^{\mathcal{C}}_{aut}(\mathsf{M})$ decidable?

**6.4.2. Data.** Tables 1, 2 and 3 group isomorphism classes of matroids $\mathsf{M}$ that have small rank and ground set based on whether $G^{\mathcal{B}}_{aut}(\mathsf{M})$ or $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are commutative or noncommutative. For numerous matroids, we were only able to determine commutativity of $G^{\mathcal{B}}_{aut}(\mathsf{M})$, and this data is recorded in tables 4 and 5. We use the matroids from `polymake`'s database `polyDB` [41], [68], which is based on the data obtained in [61]. In this database, matroids are recorded via a *reverse lexicographic basis encoding*. This encoding works in the following way. A matroid $\mathsf{M}$ with rank $r$ and ground set $\{1, \ldots, n\}$ is encoded as a binary string of length $\binom{n}{r}$. The positions of the characters in this string correspond to the $r$-element subsets of $\{1, \ldots, n\}$ in reverse lexicographic order. A `0` means that the corresponding subset is not a basis of $\mathsf{M}$, and a `1` means that the corresponding subset is a basis (recorded in this database as `*`). For brevity, we record these strings in hexadecimal and pad the beginning with `0`'s so that the string has $\lceil \frac{1}{4}\binom{n}{r} \rceil$ characters.

6.4.1. Example. Consider the Fano matroid $\mathsf{F}$ from Example 6.1.14. The bases of this matroid are all 3-element subsets of $\{1, \ldots, 7\}$ except for

$$123, 145, 246, 356, 347, 257, 167.$$

Here, $ijk$ is short for the subset $\{i, j, k\}$. There are 35 three-element subsets of $\{1, \ldots, 7\}$, and in reverse lexicographic these are

$$123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 346, 156, 256, 356, 456,$$
$$127, 137, 237, 147, 247, 347, 157, 257, 357, 457, 167, 267, 367, 467, 567.$$

The revlex basis encoding of $\mathsf{F}$ reads

`011 1111 0111 1110 1110 1111 1101 0110 1111` and so the hexadecimal encoding is `3f7eefd6f`.

Our tables in Section 6.5 comprise the complete data for all isomorphism classes of matroids with ground set sizes $n \leq 6$ and rank $r \leq 2$. We have partial data for $(r, n) = (3, 6), (3, 7)$. The columns in each table record the following information.

(i) Matroid: the matroid $\mathsf{M}$ in hexadecimal encoding;
(ii) $|E(\mathsf{M})|$: the size of the ground set of $\mathsf{M}$;
(iii) $\mathrm{rank}(\mathsf{M})$: the rank of $\mathsf{M}$;
(iv) $\mathrm{girth}(\mathsf{M})$: the girth of the matroid;
(v) # nonbases: the number $\binom{n}{r} - |\mathcal{B}(\mathsf{M})|$ where $n = |E(\mathsf{M})|$ and $r = \mathrm{rank}(\mathsf{M})$;
(vi) $|\mathrm{Aut}(\mathsf{M})|$: the order of the classical automorphism group of $\mathsf{M}$;
(vii) $d_{\mathcal{B}}(\mathsf{M})$: degree of the Gröbner basis computed for $I(G^{\mathcal{B}}_{aut}(\mathsf{M}))$.

All our results have been obtained with the `OSCAR` implementation of the noncommutative Buchberger Algorithm 3.1.19.

Since we are dealing with a computational task which a priori is not finite, it is useful to start with matroids for which it seems likely that the computation can be finished. Our experiments suggest the following heuristics. We start as low as $n = 2$, and then we stepwise increase the number of elements in the ground set. For each fixed $n$ we sort the matroids by increasing rank $r$. The set of $(r, n)$–matroids is further sorted by taking those first for which the parameter $|\mathcal{B}| \cdot \left(\sum_{k=1}^{r} \binom{n}{k} - |\mathcal{B}|\right)$ is small; the latter number is the number of relations defining the ideal $I(G^{\mathcal{B}}_{aut}(\mathsf{M}))$; cf. (6.2.1). We stopped the calculation as soon as the elapsed time exceeded one week per matroid. This resulted in a total of 207 matroids for which we could compute the commutativity of $G^{\mathcal{B}}_{aut}(\mathsf{M})$, out of which we could determine $G^{\mathcal{C}}_{aut}(\mathsf{M})$ for 85 matroids.

All computations were done on the HPC-Cluster at Technische Universität Berlin. Specifically, we used an openSUSE Leap 15.4 distribution on a cluster, where each node has 2x AMD EPYC 7302 16-core processors with 1024 gigabytes of RAM. The computation times vary a lot, matroids with $n \geq 6$ elements routinely take 24 hours or more. The total computation time for the base versions was about 33 days.

## 6.5. Tables

We distinguish whether for a given matroid $\mathsf{M}$ the quantum automorphism groups $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are commutative or not. Table 3 lists examples where $G^{\mathcal{C}}_{aut}(M) \neq G^{\mathcal{B}}_{aut}(M)$. Several uniform matroids can be spotted easily, as they are the only ones without nonbases. For instance, $G^{\mathcal{B}}_{aut}(\mathsf{U}_{2,4})$ is noncommutative but $G^{\mathcal{C}}_{aut}(\mathsf{U}_{2,4})$ is commutative (`3f` in Table 3). Our computations for the Fano matroid $\mathsf{F}$ from Example 6.1.14 exceeded the memory limit of 1024 GB; so it does not occur here.

Table 1: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are both noncommutative

| Matroid | rank($\mathsf{M}$) | $\lvert E(\mathsf{M})\rvert$ | girth($\mathsf{M}$) | #nonbases | $\lvert$ Aut($\mathsf{M}$)$\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `f` | 1 | 4 | 2 | 0 | 24 | 3 |
| `1f` | 1 | 5 | 2 | 0 | 120 | 3 |
| `0f` | 1 | 5 | 1 | 1 | 24 | 3 |
| `01` | 1 | 5 | 1 | 4 | 24 | 3 |
| `3f` | 1 | 6 | 2 | 0 | 720 | 3 |
| `1f` | 1 | 6 | 1 | 1 | 120 | 3 |
| `0f` | 1 | 6 | 1 | 2 | 48 | 3 |
| `03` | 1 | 6 | 1 | 4 | 48 | 3 |
| `01` | 1 | 6 | 1 | 5 | 120 | 3 |
| `1e` | 2 | 4 | 2 | 2 | 8 | 2 |
| `1ef` | 2 | 5 | 2 | 2 | 8 | 2 |
| `036` | 2 | 5 | 1 | 6 | 8 | 2 |
| `00f` | 2 | 5 | 2 | 6 | 24 | 3 |
| `3dff` | 2 | 6 | 2 | 2 | 16 | 2 |
| `3dfe` | 2 | 6 | 2 | 3 | 48 | 3 |
| `0fdc` | 2 | 6 | 2 | 6 | 72 | 2 |
| `06cf` | 2 | 6 | 1 | 7 | 8 | 2 |
| `01ff` | 2 | 6 | 2 | 6 | 48 | 3 |
| `01fe` | 2 | 6 | 2 | 7 | 48 | 3 |
| `0066` | 2 | 6 | 1 | 11 | 16 | 2 |
| `001f` | 2 | 6 | 2 | 10 | 120 | 3 |
| `000f` | 2 | 6 | 1 | 11 | 24 | 3 |
| `0001` | 2 | 6 | 1 | 14 | 48 | 3 |
| `079e0` | 3 | 6 | 2 | 12 | 48 | 3 |
| `001ef` | 3 | 6 | 2 | 12 | 8 | 3 |
| `00036` | 3 | 6 | 1 | 16 | 8 | 3 |

Table 1: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are both noncommutative (continued)

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert \mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `0000f` | 3 | 6 | 2 | 16 | 48 | 3 |

Table 2: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ are both commutative

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert \mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `3` | 1 | 2 | 2 | 0 | 2 | 2 |
| `3` | 1 | 3 | 1 | 1 | 2 | 2 |
| `1` | 1 | 3 | 1 | 2 | 2 | 2 |
| `7` | 1 | 4 | 1 | 1 | 6 | 2 |
| `3` | 1 | 4 | 1 | 2 | 4 | 2 |
| `1` | 1 | 4 | 1 | 3 | 6 | 2 |
| `07` | 1 | 5 | 1 | 2 | 12 | 2 |
| `03` | 1 | 5 | 1 | 3 | 12 | 2 |
| `07` | 1 | 6 | 1 | 3 | 36 | 2 |
| `1f` | 2 | 4 | 2 | 1 | 4 | 2 |
| `0b` | 2 | 4 | 1 | 3 | 6 | 2 |
| `07` | 2 | 4 | 2 | 3 | 6 | 2 |
| `03` | 2 | 4 | 1 | 4 | 2 | 2 |
| `01` | 2 | 4 | 1 | 5 | 4 | 2 |
| `1ff` | 2 | 5 | 2 | 1 | 12 | 2 |
| `07f` | 2 | 5 | 2 | 3 | 12 | 2 |
| `07e` | 2 | 5 | 2 | 4 | 12 | 2 |
| `037` | 2 | 5 | 1 | 5 | 4 | 2 |
| `013` | 2 | 5 | 1 | 7 | 12 | 2 |
| `007` | 2 | 5 | 1 | 7 | 6 | 2 |

Table 2: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ and $G^{\mathcal{C}}_{aut}(\mathsf{M})$ both commutative (continued)

| Matroid | rank($\mathsf{M}$) | $\lvert E(\mathsf{M})\rvert$ | girth($\mathsf{M}$) | #nonbases | $\lvert$ Aut($\mathsf{M}$)$\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `003` | 2 | 5 | 1 | 8 | 4 | 2 |
| `001` | 2 | 5 | 1 | 9 | 12 | 2 |
| `0fff` | 2 | 6 | 2 | 3 | 36 | 2 |
| `0fdf` | 2 | 6 | 2 | 4 | 12 | 2 |
| `06ef` | 2 | 6 | 1 | 6 | 12 | 2 |
| `00ef` | 2 | 6 | 1 | 8 | 12 | 2 |
| `00ee` | 2 | 6 | 1 | 9 | 12 | 2 |
| `0067` | 2 | 6 | 1 | 10 | 8 | 2 |
| `0023` | 2 | 6 | 1 | 12 | 36 | 2 |
| `0007` | 2 | 6 | 1 | 12 | 12 | 2 |
| `0003` | 2 | 6 | 1 | 13 | 12 | 2 |
| `07df3` | 3 | 6 | 2 | 8 | 16 | 3 |
| `079e3` | 3 | 6 | 2 | 10 | 8 | 3 |
| `02cb3` | 3 | 6 | 1 | 12 | 8 | 3 |
| `01c7e` | 3 | 6 | 2 | 11 | 36 | 3 |
| `00c36` | 3 | 6 | 1 | 14 | 12 | 3 |
| `00413` | 3 | 6 | 1 | 16 | 48 | 3 |
| `001ff` | 3 | 6 | 2 | 11 | 12 | 3 |
| `0007f` | 3 | 6 | 2 | 13 | 12 | 3 |
| `0007e` | 3 | 6 | 2 | 14 | 12 | 3 |
| `00037` | 3 | 6 | 1 | 15 | 4 | 3 |
| `00013` | 3 | 6 | 1 | 17 | 12 | 3 |
| `00007` | 3 | 6 | 1 | 17 | 12 | 3 |
| `00003` | 3 | 6 | 1 | 18 | 8 | 3 |
| `00001` | 3 | 6 | 1 | 19 | 36 | 3 |

Table 3: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is noncommutative but $G^{\mathcal{C}}_{aut}(\mathsf{M})$ is commutative

| Matroid | rank($\mathsf{M}$) | $\lvert E(\mathsf{M})\rvert$ | girth($\mathsf{M}$) | #nonbases | $\lvert \mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `3f` | 2 | 4 | 3 | 0 | 24 | 3 |
| `3ff` | 2 | 5 | 3 | 0 | 120 | 3 |
| `0b7` | 2 | 5 | 1 | 4 | 24 | 3 |
| `7fff` | 2 | 6 | 3 | 0 | 720 | 3 |
| `3fff` | 2 | 6 | 2 | 1 | 48 | 3 |
| `16ef` | 2 | 6 | 1 | 5 | 120 | 3 |
| `0267` | 2 | 6 | 1 | 9 | 48 | 3 |
| `0fff0` | 3 | 6 | 2 | 8 | 48 | 3 |
| `003ff` | 3 | 6 | 3 | 10 | 120 | 3 |
| `000b7` | 3 | 6 | 1 | 14 | 24 | 3 |

The remaining two tables list those matroids for which we could only compute either $G^{\mathcal{B}}_{aut}(\mathsf{M})$ or $G^{\mathcal{C}}_{aut}(\mathsf{M})$. Larger examples would require more than 1024 GB main memory.

Table 4: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is commutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert\mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `1` | 0 | 1 | 1 | 0 | 1 | 2 |
| `7` | 1 | 3 | 2 | 0 | 6 | 2 |
| `03f7ff` | 2 | 7 | 2 | 4 | 24 | 2 |
| `003bdf` | 2 | 7 | 1 | 9 | 36 | 2 |
| `003b9f` | 2 | 7 | 1 | 10 | 12 | 2 |
| `0019cf` | 2 | 7 | 1 | 12 | 24 | 2 |
| `0001cf` | 2 | 7 | 1 | 14 | 24 | 2 |
| `0001ce` | 2 | 7 | 1 | 15 | 24 | 2 |
| `0000c7` | 2 | 7 | 1 | 16 | 24 | 2 |
| `000007` | 2 | 7 | 1 | 18 | 36 | 2 |
| `f` | 3 | 4 | 4 | 0 | 24 | 3 |
| `fffff` | 3 | 6 | 4 | 0 | 720 | 5 |
| `7ffff` | 3 | 6 | 3 | 1 | 36 | 3 |
| `7fffe` | 3 | 6 | 3 | 2 | 72 | 3 |
| `7efff` | 3 | 6 | 3 | 2 | 8 | 3 |
| `7efdf` | 3 | 6 | 3 | 3 | 6 | 3 |
| `7efdd` | 3 | 6 | 3 | 4 | 24 | 3 |
| `37dff` | 3 | 6 | 2 | 4 | 48 | 3 |
| `379ff` | 3 | 6 | 2 | 5 | 12 | 3 |
| `12cb7` | 3 | 6 | 1 | 10 | 120 | 4 |
| `0fff7` | 3 | 6 | 3 | 5 | 12 | 3 |
| `07dff` | 3 | 6 | 2 | 6 | 8 | 3 |
| `07dfd` | 3 | 6 | 2 | 7 | 4 | 3 |

Table 4: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is commutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$ (continued)

| Matroid | rank(M) | $\|E(\mathsf{M})\|$ | girth(M) | #nonbases | $\|\mathrm{Aut}(\mathsf{M})\|$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `02cb7` | 3 | 6 | 1 | 11 | 12 | 3 |
| `01c7f` | 3 | 6 | 2 | 10 | 36 | 3 |
| `00c37` | 3 | 6 | 1 | 13 | 12 | 3 |
| `0165b96ef` | 3 | 7 | 1 | 16 | 36 | 3 |
| `0165b96ee` | 3 | 7 | 1 | 17 | 72 | 3 |
| `0165996ef` | 3 | 7 | 1 | 17 | 8 | 3 |
| `0165996af` | 3 | 7 | 1 | 18 | 6 | 3 |
| `0165996ad` | 3 | 7 | 1 | 19 | 24 | 3 |
| `0061b86ef` | 3 | 7 | 1 | 19 | 48 | 3 |
| `0061b06ef` | 3 | 7 | 1 | 20 | 12 | 3 |
| `002098267` | 3 | 7 | 1 | 25 | 240 | 4 |
| `000ffbfe0` | 3 | 7 | 2 | 17 | 24 | 3 |
| `000f7bde7` | 3 | 7 | 2 | 16 | 8 | 3 |
| `0005b96e7` | 3 | 7 | 1 | 20 | 12 | 3 |
| `0003f8fff` | 3 | 7 | 2 | 16 | 24 | 3 |
| `0003f8ffd` | 3 | 7 | 2 | 17 | 12 | 3 |
| `0003f8fe3` | 3 | 7 | 2 | 19 | 48 | 5 |
| `0003f0fdf` | 3 | 7 | 2 | 18 | 24 | 3 |
| `0003f0fdc` | 3 | 7 | 2 | 20 | 24 | 3 |
| `0003f0fc3` | 3 | 7 | 2 | 21 | 24 | 3 |
| `0001b86ef` | 3 | 7 | 1 | 21 | 8 | 3 |
| `0001b86ed` | 3 | 7 | 1 | 22 | 4 | 3 |
| `0001b86e3` | 3 | 7 | 1 | 23 | 16 | 3 |
| `0001b06c3` | 3 | 7 | 1 | 25 | 8 | 3 |
| `000098267` | 3 | 7 | 1 | 26 | 24 | 3 |

Table 4: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is commutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$ (continued)

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert\mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `000098263` | 3 | 7 | 1 | 27 | 16 | 3 |
| `0000380ef` | 3 | 7 | 1 | 25 | 36 | 3 |
| `0000380ee` | 3 | 7 | 1 | 26 | 36 | 3 |
| `000018067` | 3 | 7 | 1 | 28 | 24 | 3 |
| `000018066` | 3 | 7 | 1 | 29 | 24 | 3 |
| `000008023` | 3 | 7 | 1 | 31 | 144 | 3 |
| `000000fff` | 3 | 7 | 2 | 23 | 36 | 3 |
| `000000fdf` | 3 | 7 | 2 | 24 | 12 | 3 |
| `0000006ef` | 3 | 7 | 1 | 26 | 12 | 3 |
| `0000000ef` | 3 | 7 | 1 | 28 | 12 | 3 |
| `0000000ee` | 3 | 7 | 1 | 29 | 12 | 3 |
| `000000067` | 3 | 7 | 1 | 30 | 8 | 3 |
| `000000023` | 3 | 7 | 1 | 32 | 36 | 3 |
| `000000007` | 3 | 7 | 1 | 32 | 24 | 3 |
| `000000003` | 3 | 7 | 1 | 33 | 24 | 3 |

Table 5: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is noncommutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert\mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `7f` | 1 | 7 | 2 | 0 | 5040 | 3 |
| `3f` | 1 | 7 | 1 | 1 | 720 | 3 |
| `1f` | 1 | 7 | 1 | 2 | 240 | 3 |
| `0f` | 1 | 7 | 1 | 3 | 144 | 3 |
| `07` | 1 | 7 | 1 | 4 | 144 | 3 |

Table 5: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is noncommutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$ (continued)

| Matroid | rank(M) | $\|E(\mathsf{M})\|$ | girth(M) | #nonbases | $\|\mathrm{Aut}(\mathsf{M})\|$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| 03 | 1 | 7 | 1 | 5 | 240 | 3 |
| 1fffff | 2 | 7 | 3 | 0 | 5040 | 3 |
| 0fffff | 2 | 7 | 2 | 1 | 240 | 3 |
| 0f7fff | 2 | 7 | 2 | 2 | 48 | 2 |
| 0f7fbf | 2 | 7 | 2 | 3 | 48 | 3 |
| 05bbdf | 2 | 7 | 1 | 6 | 720 | 3 |
| 03ffff | 2 | 7 | 2 | 3 | 144 | 3 |
| 03f7fe | 2 | 7 | 2 | 5 | 48 | 2 |
| 03f73f | 2 | 7 | 2 | 6 | 72 | 2 |
| 01bbdf | 2 | 7 | 1 | 7 | 48 | 3 |
| 01b3df | 2 | 7 | 1 | 8 | 16 | 2 |
| 01b3de | 2 | 7 | 1 | 9 | 48 | 3 |
| 0099cf | 2 | 7 | 1 | 11 | 240 | 3 |
| 007fff | 2 | 7 | 2 | 6 | 144 | 3 |
| 007fbf | 2 | 7 | 2 | 7 | 48 | 3 |
| 007fbc | 2 | 7 | 2 | 9 | 144 | 3 |
| 003b9c | 2 | 7 | 1 | 12 | 72 | 2 |
| 00198f | 2 | 7 | 1 | 13 | 16 | 2 |
| 0008c7 | 2 | 7 | 1 | 15 | 144 | 3 |
| 0007ff | 2 | 7 | 2 | 10 | 240 | 3 |
| 0007fe | 2 | 7 | 2 | 11 | 240 | 3 |
| 0003df | 2 | 7 | 1 | 12 | 48 | 3 |
| 0003de | 2 | 7 | 1 | 13 | 48 | 3 |
| 0000c6 | 2 | 7 | 1 | 17 | 48 | 2 |
| 000043 | 2 | 7 | 1 | 18 | 144 | 3 |

Table 5: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is noncommutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$ (continued)

| Matroid | rank(M) | $\lvert E(\mathsf{M})\rvert$ | girth(M) | #nonbases | $\lvert\mathrm{Aut}(\mathsf{M})\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `00003f` | 2 | 7 | 2 | 15 | 720 | 3 |
| `00001f` | 2 | 7 | 1 | 16 | 120 | 3 |
| `00000f` | 2 | 7 | 1 | 17 | 48 | 3 |
| `000003` | 2 | 7 | 1 | 19 | 48 | 3 |
| `000001` | 2 | 7 | 1 | 20 | 240 | 3 |
| `0ffff` | 3 | 6 | 3 | 4 | 48 | 3 |
| `079ef` | 3 | 6 | 2 | 8 | 16 | 3 |
| `07fffffff` | 3 | 7 | 3 | 4 | 144 | 3 |
| `00e3f0fdc` | 3 | 7 | 2 | 17 | 144 | 3 |
| `000f7bde0` | 3 | 7 | 2 | 19 | 16 | 3 |
| `0005b96ef` | 3 | 7 | 1 | 19 | 48 | 3 |
| `0005b96e0` | 3 | 7 | 1 | 23 | 48 | 3 |
| `0003f0fc0` | 3 | 7 | 2 | 23 | 48 | 4 |
| `0001b06cf` | 3 | 7 | 1 | 23 | 16 | 3 |
| `0001b06c0` | 3 | 7 | 1 | 27 | 48 | 3 |
| `0000781ff` | 3 | 7 | 2 | 22 | 144 | 3 |
| `0000781fe` | 3 | 7 | 2 | 23 | 144 | 3 |
| `000007fff` | 3 | 7 | 3 | 20 | 720 | 3 |
| `000003fff` | 3 | 7 | 2 | 21 | 48 | 3 |
| `000003dff` | 3 | 7 | 2 | 22 | 16 | 3 |
| `000003dfe` | 3 | 7 | 2 | 23 | 48 | 3 |
| `0000016ef` | 3 | 7 | 1 | 25 | 120 | 3 |
| `000000fdc` | 3 | 7 | 2 | 26 | 72 | 3 |
| `0000006cf` | 3 | 7 | 1 | 27 | 8 | 3 |
| `000000267` | 3 | 7 | 1 | 29 | 48 | 3 |

Table 5: $G^{\mathcal{B}}_{aut}(\mathsf{M})$ is noncommutative but we have no data on $G^{\mathcal{C}}_{aut}(\mathsf{M})$ (continued)

| Matroid | rank($\mathsf{M}$) | $\lvert E(\mathsf{M})\rvert$ | girth($\mathsf{M}$) | #nonbases | $\lvert$ Aut($\mathsf{M}$)$\rvert$ | $d_{\mathcal{B}}(\mathsf{M})$ |
|---|---|---|---|---|---|---|
| `0000001ff` | 3 | 7 | 2 | 26 | 48 | 3 |
| `0000001fe` | 3 | 7 | 2 | 27 | 48 | 3 |
| `000000066` | 3 | 7 | 1 | 31 | 16 | 3 |
| `00000001f` | 3 | 7 | 2 | 30 | 240 | 3 |
| `00000000f` | 3 | 7 | 1 | 31 | 48 | 3 |
| `000000001` | 3 | 7 | 1 | 34 | 144 | 3 |

## 6.6. Lovász's Theorem for Matroids

Throughout this chapter our aim was to introduce and study interesting new classes of noncommutative algebras, namely the various quantum automorphism groups of matroids. But it is natural to ask if we can also learn something about matroids from their quantum automorphisms. While we currently lack the technology to obtain a deep result in that direction, here we want to sketch an idea that we find intriguing.

Our point of departure is a celebrated result in graph theory. Lovász in [50] derives the following characterisation of isomorphic graphs in terms of graph homomorphism counts.

6.6.1. Theorem. *Two graphs $\Gamma_1$ and $\Gamma_2$, without multiple edges, are isomorphic if and only if*

$$|\operatorname{Hom}(\Gamma, \Gamma_1)| = |\operatorname{Hom}(\Gamma, \Gamma_2)|$$

*for all graphs $\Gamma$.*

Exciting recent work of Mancinska and Roberson [55] characterises *quantum* isomorphic graphs in terms of homomorphism counts from *planar* graphs.

6.6.2. Theorem. *Two graphs $\Gamma_1$ and $\Gamma_2$, without multiple edges, are quantum isomorphic if and only if*

$$|\operatorname{Hom}(\Gamma, \Gamma_1)| = |\operatorname{Hom}(\Gamma, \Gamma_2)|$$

*for all planar graphs $\Gamma$.*

We want to derive a matroidal analog of Theorem 6.6.1, which is the first critical step to see if *quantum isomorphic* matroids admit a characterisation in terms of counts of maps of matroids. Matroid theory is somewhat complicated from a categorical point of view. Namely, there are at least two competing concepts for morphisms, which are discussed and compared in Chapters 8 and 9 of [85]. One type of such morphisms are the strong maps, which we will define next. Let $\mathsf{M}_1$ and $\mathsf{M}_2$ be matroids. A *strong map* $\varphi : \mathsf{M}_1 \to \mathsf{M}_2$ is a function

$$\varphi : E(\mathsf{M}_1) \sqcup \{e\} \to E(\mathsf{M}_2) \sqcup \{e\}$$

such that $\varphi(e) = e$ and the inverse image of a flat of $\mathsf{M}_2 \oplus \mathsf{U}_{0,e}$ is a flat of $\mathsf{M}_1 \oplus \mathsf{U}_{0,e}$. (Here, $\mathsf{U}_{0,e}$ is the matroid consisting of a single loop element $e$, see Example 6.1.13.) The *image* of $\varphi$, denoted by $\varphi(\mathsf{M}_1)$, is the restriction $\mathsf{M}_2|_{E_1'}$ where $E_1' = \varphi(E(\mathsf{M}_1) \cup \{e\}) \cap E(\mathsf{M}_2)$. Given $L \subseteq E(\mathsf{M})$, the *restriction* $\mathsf{M}_{|L}$ is simply the deletion of $E(\mathsf{M}) \setminus L$ from $\mathsf{M}$ as defined at the beginning of §6.3. Denote by $\operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2)$ the set of strong maps $\varphi : \mathsf{M}_1 \to \mathsf{M}_2$. An *embedding* of matroids is a strong map $\varphi : \mathsf{M}_1 \hookrightarrow \mathsf{M}_2$ such that $\varphi : E(\mathsf{M}_1) \sqcup \{e\} \to E(\mathsf{M}_2) \sqcup \{e\}$ is injective and $\mathsf{M}_1 \cong \varphi(\mathsf{M}_1)$. We are now equipped for the main result of this section.

6.6.3. THEOREM. *Two matroids* $\mathsf{M}_1$ *and* $\mathsf{M}_2$ *are isomorphic to each other if and only if*

$$|\operatorname{Hom}(\mathsf{M}_1, \mathsf{L})| = |\operatorname{Hom}(\mathsf{M}_2, \mathsf{L})|$$

*for all matroids* $\mathsf{L}$.

Note that the crucial difference between Theorems 6.6.1 and 6.6.3 is that Theorem 6.6.1 concerns counts of graph homomorphisms *to* the graphs $\Gamma_1$ and $\Gamma_2$, whereas Theorem 6.6.3 concerns counts of strong maps *from* the matroids $\mathsf{M}_1$ and $\mathsf{M}_2$. This is reflected in the fact that if $\varphi : \Gamma_1 \to \Gamma_2$ is a graph homomorphism between graphs with the same number of vertices such that $\varphi(\Gamma_1) = \Gamma_2$, then $\varphi$ is an isomorphism. However, if $\psi : \mathsf{M}_1 \to \mathsf{M}_2$ is an embedding of matroids on ground sets of the same size, then $\psi$ is an isomorphism.

6.6.4. LEMMA. *The matroids* $\mathsf{M}_1$ *and* $\mathsf{M}_2$ *are isomorphic if and only if there are surjective strong maps* $\varphi_1 : \mathsf{M}_1 \to \mathsf{M}_2$ *and* $\varphi_2 : \mathsf{M}_2 \to \mathsf{M}_1$.

PROOF. Since the maps $\varphi_1 : E(\mathsf{M}_1) \sqcup \{e\} \to E(\mathsf{M}_2) \sqcup \{e\}$ and $\varphi_2 : E(\mathsf{M}_2) \sqcup \{e\} \to E(\mathsf{M}_1) \sqcup \{e\}$ are both surjective maps of finite sets, they are bijective. Since $\varphi_1$ and $\varphi_2$ are strong maps of the same size, the matroids $\mathsf{M}_1$ and $\mathsf{M}_2$ have the same rank, and hence $\varphi_1$ and $\varphi_2$ define isomorphisms, see [46, Proposition 8.1.6]. $\square$

Denote by $\operatorname{Emb}(\mathsf{M}_1, \mathsf{M}_2)$ the set of embeddings $\mathsf{M}_1 \hookrightarrow \mathsf{M}_2$ and $\operatorname{Surj}(\mathsf{M}_1, \mathsf{M}_2)$ the set of surjective strong maps $\mathsf{M}_1 \to \mathsf{M}_2$, respectively. Given another matroid $\mathsf{N}$, set

$$\operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N}) = \{\varphi \in \operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2) \,:\, \varphi(\mathsf{M}_1) \cong \mathsf{N}\}.$$

The following lemma and its proof are matroidal analogs of [50, Equation 6].

6.6.5. LEMMA. *Given matroids* $\mathsf{M}_1$ *and* $\mathsf{M}_2$*, we have*

$$|\operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2)| = \sum_{\mathsf{N} \in \mathcal{M}} |\operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})| = \sum_{\mathsf{N} \in \mathcal{M}} \frac{|\operatorname{Surj}(\mathsf{M}_1, \mathsf{N})||\operatorname{Emb}(\mathsf{N}, \mathsf{M}_2)|}{|\operatorname{Aut}(\mathsf{N})|}$$

*where* $\mathcal{M}$ *is a set of representatives of the isomorphism classes of matroids.*

PROOF. Given $\varphi \in \operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})$, define

$$E_\varphi = \{(\pi, \psi) \,:\, \pi \in \operatorname{Surj}(\mathsf{M}_1, \mathsf{N}),\ \psi \in \operatorname{Emb}(\mathsf{N}, \mathsf{M}_2),\ \varphi = \psi \circ \pi\}$$

We claim the following:

(i) $|E_\varphi| = |\operatorname{Aut}(\mathsf{N})|$;
(ii) $E_{\varphi_1} \cap E_{\varphi_2} = \emptyset$ for different $\varphi_1, \varphi_2 \in \operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})$;
(iii) every pair $(\pi, \psi)$ with $\pi \in \operatorname{Surj}(\mathsf{M}_1, \mathsf{N})$ and $\psi \in \operatorname{Emb}(\mathsf{N}, \mathsf{M}_2)$ is in some $E_\varphi$.

Properties (ii) and (iii) are clear, so consider (i). First, we claim that $E_\varphi \neq \emptyset$. By [46, Lemma 8.1.4], we can factor $\varphi$ into strong maps $\pi' : \mathsf{M}_1 \to \varphi(\mathsf{M}_1)$ and $\psi' : \varphi(\mathsf{M}_1) \to \mathsf{M}_2$. Fix an isomorphism $\sigma : \varphi(\mathsf{M}_1) \to \mathsf{N}$. Then $(\sigma \circ \pi', \psi' \circ \sigma^{-1}) \in \operatorname{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})$.

Fix a pair $(\pi_0, \psi_0) \in E_\varphi$ and set

$$E'_\varphi = \{(\sigma \circ \pi_0, \psi_0 \circ \sigma^{-1}) \,:\, \sigma \in \mathrm{Aut}(\mathsf{N})\}$$

If $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathsf{N})$ are distinct, then $\sigma_1 \circ \pi_0 \neq \sigma_2 \circ \pi_0$ since $\pi_0$ is surjective. So $|E'_\varphi| = |\mathrm{Aut}(\mathsf{N})|$ and $E'_\varphi \subseteq E_\varphi$. Conversely, suppose $(\pi, \psi) \in E_\varphi$. The images $\psi(\mathsf{N})$ and $\psi_0(\mathsf{N})$ both equal $\varphi(\mathsf{M}_1)$, and $\psi, \psi_0 : \mathsf{N} \to \varphi(\mathsf{M}_1)$ are isomorphisms. Then $\sigma = \psi^{-1} \circ \psi_0$ is an automorphism of $\mathsf{N}$ and $(\pi, \psi) = (\sigma \circ \pi_0, \psi_0 \circ \sigma^{-1})$. Therefore $E_\varphi = E'_\varphi$, which proves (i).

By these properties, $\{E_\varphi \,:\, \varphi \in \mathrm{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})\}$ form a partition of a set of size $|\mathrm{Surj}(\mathsf{M}_1, \mathsf{N})||\mathrm{Emb}(\mathsf{N}, \mathsf{M}_2)|$ into $|\mathrm{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})|$ subsets of size $|\mathrm{Aut}(\mathsf{N})|$. So

$$|\mathrm{Hom}(\mathsf{M}_1, \mathsf{M}_2; \mathsf{N})||\mathrm{Aut}(\mathsf{N})| = |\mathrm{Surj}(\mathsf{M}_1, \mathsf{N})||\mathrm{Emb}(\mathsf{N}, \mathsf{M}_2)|,$$

from which the theorem follows. $\square$

Proof of Theorem 6.6.3. We prove that $|\mathrm{Surj}(\mathsf{M}_1, \mathsf{L})| = |\mathrm{Surj}(\mathsf{M}_2, \mathsf{L})|$ for all matroids $\mathsf{L}$. If $\mathsf{L} = \mathsf{M}_2$ then $\mathrm{Surj}(\mathsf{M}_1, \mathsf{M}_2) \neq \emptyset$, and if $\mathsf{L} = \mathsf{M}_1$ then $\mathrm{Surj}(\mathsf{M}_2, \mathsf{M}_1) \neq \emptyset$. The theorem then follows from Lemma 6.6.4.

We proceed by induction on $|E(\mathsf{L})|$. The equality $|\mathrm{Surj}(\mathsf{M}_1, \mathsf{L})| = |\mathrm{Surj}(\mathsf{M}_2, \mathsf{L})|$ is clear when $|E(\mathsf{L})| = 1$. Now, observe that

$$|\mathrm{Surj}(\mathsf{M}_i, \mathsf{L})| = \sum_{\substack{\mathsf{N} \in \mathcal{M} \\ |E(\mathsf{N})| = |E(\mathsf{L})|}} |\mathrm{Hom}(\mathsf{M}_i, \mathsf{L}; \mathsf{N})|$$

By Lemma 6.6.5, we have

$$|\mathrm{Hom}(\mathsf{M}_i, \mathsf{L})| = |\mathrm{Surj}(\mathsf{M}_i, \mathsf{L})| + \sum_{\substack{\mathsf{N} \in \mathcal{M} \\ |E(\mathsf{N})| < |E(\mathsf{L})|}} \frac{|\mathrm{Surj}(\mathsf{M}_i, \mathsf{N})||\mathrm{Emb}(\mathsf{N}, \mathsf{L})|}{|\mathrm{Aut}(\mathsf{N})|}.$$

Since $|\mathrm{Hom}(\mathsf{M}_1, \mathsf{L})| = |\mathrm{Hom}(\mathsf{M}_2, \mathsf{L})|$, the difference $|\mathrm{Surj}(\mathsf{M}_1, \mathsf{L})| - |\mathrm{Surj}(\mathsf{M}_2, \mathsf{L})|$ is

$$\sum_{\substack{\mathsf{N} \in \mathcal{M} \\ |E(\mathsf{N})| < |E(\mathsf{L})|}} \big(|\mathrm{Surj}(\mathsf{M}_2, \mathsf{N})| - |\mathrm{Surj}(\mathsf{M}_1, \mathsf{N})|\big) \frac{|\mathrm{Emb}(\mathsf{N}, \mathsf{L})|}{|\mathrm{Aut}(\mathsf{N})|}.$$

This equals 0 by the inductive hypothesis. $\square$

6.6.6. Remark. We briefly sketch the following alternative argument for Theorem 6.6.3, suggested to us by A. Chirvasitu. The idea is to consider the category of matroids $\mathfrak{M}$ whose morphisms are strong maps. Surjective maps then reflect the epimorphisms of the category and injective maps the monomorphisms; cf. Propositions 3.4 and 3.7 in [43]. Since we are considering only matroids with a finite ground set, the set of morphisms between two matroids is finite, and it follows that there are only finitely many injective maps into a given matroid. Both facts imply that $\mathfrak{M}$ is quasifinite and locally finite. Proposition 5.4 in [43] shows that $\mathfrak{M}$ has equalisers, which implies that the images of morphisms are such that every quotient is an epimorphism. Theorem 2.2 in [70], which states that every locally finite, quasifinite category whose images are such that every quotient is an isomorphism is combinatorial, then directly recovers the Theorem 6.6.3.

The above discussion inspires new questions around the subject of quantum isomorphisms of matroids. First, how can one define such a form of isomorphism? Does it, similar to quantum automorphism groups of matroids, have to depend on the choice of axiom system, or can one find a definition, that is independent of the axiom system?

Once this is defined one can also ask, inspired by Theorem 6.6.3 and Theorem 6.6.2, whether a statement similar to Theorem 6.6.2 also exists for matroids.

6.6.7. Question. For a sensible definition of quantum isomorphism of matroids, does a class of matroids $\mathcal{L}$ exist such that two matroids $\mathsf{M}_1$ and $\mathsf{M}_2$ are quantum isomorphic if and only if

$$|\operatorname{Hom}(\mathsf{M}_1, \mathsf{L})| = |\operatorname{Hom}(\mathsf{M}_2, \mathsf{L})|$$

for all matroids $\mathsf{L} \in \mathcal{L}$?

CHAPTER 7

# Open Questions

In this chapter, we collect some open questions on the topics covered in this thesis.

## 7.1. Existence of Families of Graphs with Restrictions on Automorphism Groups for Quantum Symmetries

7.1.1. QUESTION. For which families of graphs $\mathcal{F}$ does a group $G$ exist such that if a graph $\Gamma$ is in $\mathcal{F}$ and has classical automorphism group $G$, then $\Gamma$ does not have quantum symmetries?

Due to the result of van Dobben de Bruyn, Roberson and Schmidt in [31] that there exists a graph with quantum symmetry but with trivial automorphism group, we know that when taking $\mathcal{F}$ to be all simple graphs there exist no groups that satisfy the above condition: calling the graph constructed in [31] $\Gamma_0$ and given a graph $\Gamma$ with any automorphism group, one could simply take the disjoint union of $\Gamma$ and $\Gamma_0$ and would get a graph with the same automorphism group as $\Gamma$ (as long as $\Gamma$ is not the same as $\Gamma_0$) that now has quantum symmetries. In fact, in their paper they even state a more general result, namely that for any given group, there exist both graphs with that group as an automorphism group that do have quantum symmetries and graphs that do not have quantum symmetries.

On the other hand, in [39], Fulton shows that when $\mathcal{F}$ is the class of all trees, then for $G = \mathbb{Z}_2$ no graph $\Gamma \in \mathcal{F}$ has quantum symmetries.

The question is now for which other classes of graphs there exist groups that prohibit the existence of quantum symmetries. One could for example, inspired by Observation 2, study vertex-transitive graphs, since the existence of an asymmetric graph with quantum symmetries can not be used to take any vertex-transitive graph and add quantum symmetries while staying vertex-transitive.

## 7.2. Quantum Symmetries of Vertex-Transitive Graphs

In Chapter 4 the quantum automorphism groups for most vertex-transitive graphs on 12 vertices could be computed, however for some it still remains unknown.

7.2.1. QUESTION. What are the quantum automorphism groups of the remaining vertex-transitive graphs on 12 vertices, i.e. of $K_2\Box C_6(2)$, $C_{12}(5)$, $C_{12}(5,6)$, $C_{12}(5^+)$ and of $C_{12}(5^+,6)$?

Taking the classical automorphism groups of these graphs as a first lead, one could for example guess that the quantum automorphism group of $K_2\Box C_6(2)$ is $\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr_* S_3)$ or that the quantum automorphism group of $C_{12}(5^+)$ or of $C_{12}(5^+, 6)$ is $H_3^+$ and try to show that.

Another immediate question are the quantum symmetries of the vertex-transitive graphs on more vertices.

7.2.2. Question. What are the quantum automorphism groups of vertex-transitive graphs on 14, 15 or 16 vertices? Alternatively, which of those vertex-transitive graphs have quantum symmetries?

A lot of the techniques used in this thesis should still be applicable to find the answer to the above question, they might however be more tedious to apply, since with the number of vertices also the number of cases to check increases.

### 7.3. Questions on Quantum Switching Isomorphisms

In Chapter 5 we found that for connected signed graphs, any quantum switching isomorphism must always come from a quantum isomorphism of the delabelled graphs. But what about non-connected signed graphs?

7.3.1. Question. If we have two non-connected signed graphs, can they be quantum switching isomorphic without the delabelled graphs being quantum isomorphic? Or does the same restriction as for connected graphs hold?

Another question one can ask is what about more generalisations of complex reflection groups? In [3] a free quantum version $H_n^{s+}$ of the group $H_n^s = \mathbb{Z}_s \wr S_n$ was introduced and in [11] it was studied further. For the case $s = 1$ we have that $H_n^{s+} = S_n^+$, which leads to the original quantum isomorphism of graphs. When $s = 2$, we get $H_n^{s+} = H_n^+$, which leads to the quantum switching isomorphism we studied in this thesis. But what about $s > 2$?

7.3.2. Question. Can we define a kind of quantum isomorphism of labelled graphs with $s$ edge-labels that relates to the quantum group $H_n^{s+}$ in a similar way that quantum isomorphism relates to $S_n^+$ and quantum switching isomorphism relates to $H_n^+$?

### 7.4. Questions on quantum automorphisms of matroids

In the chapter about quantum automorphisms of matroids, already several questions were introduced. We collect them again here.

7.4.1. Question. Does the statement of Theorem 6.2.7 hold for any matroid, i.e. do we have that for any matroid $\mathsf{M}$, it holds

$$G_{aut}^{\mathcal{C}}(\mathsf{M}) \leq G_{aut}^{\mathcal{B}}(\mathsf{M})?$$

The next question is similar to the first question in this chapter, as it concerns the decidability of the existence of quantum symmetries.

7.4.2. Question. Given a matroid $\mathsf{M}$, is the commutativity of $G^{\mathcal{B}}_{aut}(\mathsf{M})$ or $G^{\mathcal{C}}_{aut}(\mathsf{M})$ decidable?

Inspired by quantum isomorphisms of graphs, one can ask whether a similar notion also exists for matroids. However, since the quantum automorphism groups of matroids depend on the chosen axiom system, it is not immediately clear, whether there should be only one such quantum isomorphism of matroids or several, again depending on the axiom system.

7.4.3. Question. Can one define quantum isomorphisms of matroids? And can it be done in a way that is independent of the axiom system?

The last question concerning matroids is inspired by the result in [55] that quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. Since a Lovász type theorem exists for classical isomorphism and equality of strong homomorphism counts to matroids, the question whether such a statement for a kind of quantum isomorphism of matroids also holds true poses itself:

7.4.4. Question. Does a family $\mathcal{L}$ of matroids exist, such that two matroids $\mathsf{M}_1$ and $\mathsf{M}_2$ are quantum isomorphic if and only if

$$|\operatorname{Hom}(\mathsf{M}_1, \mathsf{L})| = |\operatorname{Hom}(\mathsf{M}_2, \mathsf{L})|$$

for all matroids $\mathsf{L} \in \mathcal{L}$?

And if quantum isomorphisms of matroids depend on the axiom system, does the family $\mathcal{L}$ also depend on this axiom system or can it even be chosen independently?

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